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Belief Revision with General Epistemic States

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Abstract

In order to properly regulate iterated belief revision, Darwiche and Pearl (1997) model belief revision as revising epistemic states by propositions. An epistemic state in their sense consists of a belief set and a set of conditional beliefs. Although the denotation of an epistemic state can be indirectly captured by a total preorder on the set of worlds, it is unclear how to *directly* capture the structure in terms of the beliefs and conditional beliefs it contains. In this paper, we first provide an axiomatic characterisation for epistemic states by using nine rules about beliefs and conditional beliefs, and then argue that the last two rules are too strong and should be eliminated for characterising the belief state of an agent. We call a structure which satisfies the first seven rules a *general epistemic state* (GEP). To provide a semantical characterisation of GEPs, we introduce a mathematical structure called *belief algebra*, which is in essence a certain binary relation defined on the power set of worlds. We then establish a 1-1 correspondence between GEPs and belief algebras, and show that total preorders on worlds are special cases of belief algebras. Furthermore, using the notion of belief algebras, we extend the classical iterated belief revision rules of Darwiche and Pearl to our setting of general epistemic states.

Introduction

Belief revision mainly characterises how an agent changes her belief with new evidence. It concerns how to represent and update an agent's belief. Logic based belief revision has been extensively studied in AI (Alchourron, Gärdenfors, and Makinson 1985; Katsuno and Mendelzon 1991; Darwiche and Pearl 1997; Jin and Thielscher 2007; Benferhat, Lagrue, and Papini 2005; Tamargo et al. 2011; Ma, Benferhat, and Liu 2012; Delgrande 2012) in the past three decades.

The AGM framework of belief revision (Alchourron, Gärdenfors, and Makinson 1985) represents an agent's current belief as a set of propositions, called a *belief set*, and represents the new evidence by a single proposition. The revision result is also represented as a belief set. Noticing that the AGM framework is too permissive to enforce iterated belief revision, Darwiche and Pearl argue that belief revision should be extended to permit operations on epistemic states, rather than belief sets (Darwiche and Pearl 1997). Each epistemic state Ψ consists of a belief set and a set of conditional beliefs, and a conditional belief $(\beta|\alpha)$ is in Ψ if and only if β is entailed by the revision of Ψ with α. Furthermore, they show that each epistemic state corresponds to a total preorder on the set of worlds. But it remains unclear how to characterise an epistemic state directly in terms of the beliefs and conditional beliefs it contains. This is rather unlike the notion of belief set, which is defined as a set of propositions that is deductively closed.

One aim of this paper is to provide an axiomatic characterisation for epistemic states. We identify nine rules about the beliefs and conditional beliefs contained in an epistemic state, and then prove that these rules are *necessary and sufficient* for characterising epistemic states. The first seven rules are quite natural. For example, one rule describes that if ϕ is a belief in Ψ, then $(\phi \mid T)$ is a conditional belief in Ψ, where T is a fixed tautology; and another rule describes that each epistemic state contains the conditional belief $(\phi \mid \phi)$ for any consistent proposition ϕ. Moreover, these rules are all *closed under intersection*, i.e. if Ψ_1 and Ψ_2 both satisfy these rules, then so does their intersection $\Psi_1 \cap \Psi_2$. However, the last two rules ((E8) and (E9)) are neither obvious nor closed under intersection. Take (E8) as an example. Suppose α, β, γ are pairwise inconsistent propositions and Ψ is an epistemic state which contains the conditional belief $(\alpha\vee\beta \mid \alpha\vee\beta\vee\gamma)$. Then (E8) specifies that Ψ should contain either $(\alpha \mid \alpha \vee \gamma)$ or $(\beta \mid \beta \vee \gamma)$. This shows that an epistemic state is always *deterministic* or *certain* when an agent has 'disjunctive' conditional beliefs.

In this paper, we propose to eliminate these two rules and call a structure which satisfies the first seven rules a *general epistemic state* (GEP). A GEP differs from an epistemic state only in the two rules (E8) and (E9). Because the absence of these two rules, we no longer can characterise the belief information contained in a GEP by a total preorder on the set of worlds. To provide a semantical characterisation of GEPs, we introduce a binary relation $\gg$ on the power set of worlds.

Suppose W is the set of worlds and Ψ is a GEP. For a proposition ϕ, we write $\mathsf{Mod}(\phi)$ for the set of worlds of ϕ. If ψ is a belief in Ψ, then we impose $\mathsf{Mod}(\psi) \gg \mathsf{Mod}(\neg\psi)$; if $(\beta \mid \alpha)$ is a conditional belief in Ψ, then we impose $\mathsf{Mod}(\alpha \wedge \beta) \gg \mathsf{Mod}(\alpha \wedge \neg\beta)$. Here, for two jointly inconsistent propositions α and β, $\mathsf{Mod}(\alpha) \gg \mathsf{Mod}(\beta)$ is interpreted as "α is more believable than β." We call the mathematical structure $(W, \gg)$ a *belief algebra* on W.

We then give an axiomatic characterisation of belief algebras and establish a 1-1 correspondence between GEPs and belief algebras. Meanwhile, we show that total preorders on worlds are special cases of belief algebras.

Furthermore, using the notion of belief algebras, we extend the classical iterated belief revision rules of Darwiche and Pearl to belief revision of a GEP by another GEP.

Notation and definitions

In this paper, we restrict our discussion to belief revision in a finite propositional language L. We denote by T a fixed tautology in L and write W for the set of all worlds (interpretations) of L. A *theory* K of L is a set of propositions which is deductively closed.

For each proposition ψ, we denote by $\mathsf{Mod}(\psi)$ the set of all worlds of ψ, i.e. $\mathsf{Mod}(\psi)$ consists of all models of ψ in L. For a subset $S \subseteq W$, we denote by $\overline{S}$ the complement of S, and by $\textsc{Form}(S)$ a proposition whose worlds are exactly those in S. Clearly, we have $\textsc{Form}(\mathsf{Mod}(\psi)) \equiv \psi$ and $\mathsf{Mod}(\textsc{Form}(S)) = S$.

A (partial) preorder $\preceq$ on A is a binary relation on A which is reflexive and transitive. A preorder $\preceq$ is called *total* if any two elements in A are comparable under $\preceq$. We write $x \sim y$ if $x \preceq y$ and $y \preceq x$, and $x \prec y$ if $x \preceq y$ but $y \not\preceq x$.

Belief revision and epistemic states

The most famous belief revision approach is AGM theory, which uses eight postulates to characterise the rules of belief revision (Alchourron, Gärdenfors, and Makinson 1985). In AGM's approach, the belief of an agent is represented by a theory in L, called a *belief set*. An AGM revision operator $\circ$ is a function mapping a belief set and a proposition to another belief set, which satisfies several postulates. Katsuno and Mendelzon (Katsuno and Mendelzon 1991) rephrase the AGM postulates in a propositional logic setting and give a representation theorem for AGM operators.

Researchers observe that belief set alone is not sufficient to uniquely determine a belief revision strategy and extra-logical factors such as conditional belief (Spohn 1988; Kern-Isberner 1999; Darwiche and Pearl 1997) and epistemic entrenchment (Gärdenfors and Makinson 1988) are proposed to overcome this shortcoming. In this paper, we will focus on conditional beliefs.

A *conditional belief* has the form $(\beta \mid \alpha)$, where α, β are propositions in L. An agent has a conditional belief $(\beta \mid \alpha)$ if she will believe β whenever she believes α. Darwiche and Pearl (Darwiche and Pearl 1997) use epistemic states to model an agent's belief states and propose a theory of belief revision based on epistemic states. An epistemic state consists of a belief set and a set of conditional beliefs. The denotation of an epistemic state Ψ is captured by the AGM operator $\circ$ defined by Ψ via (EP):

(EP) $\beta \in \Psi \circ \alpha$ iff $(\beta \mid \alpha) \in \Psi$.

Definition 1. Suppose Ψ consists of a belief set (denoted by Bel(Ψ)) and a set of conditional beliefs. We say Ψ is an epistemic state if the operator $\circ$ defined by Ψ via (EP) satisfies (R^*1)-(R^*6) below.

(R^*1) $\Psi \circ \mu$ implies μ.

(R^*2) If $\Psi \wedge \mu$ is satisfiable, then $\Psi \circ \mu \equiv \Psi \wedge \mu$.

(R^*3) If μ is satisfiable, then $\Psi \circ \mu$ is satisfiable.

(R^*4) If $\Psi_1 = \Psi_2$ and $\mu_1 \equiv \mu_2$, then $\Psi_1 \circ \mu_1 \equiv \Psi_2 \circ \mu_2$.

(R^*5) $(\Psi \circ \mu) \wedge \phi$ implies $\Psi \circ (\mu \wedge \phi)$.

(R^*6) If $(\Psi \circ \mu) \wedge \phi$ is satisfiable, then $\Psi \circ (\mu \wedge \phi)$ implies $(\Psi \circ \mu) \wedge \phi$.

We note that in the above rules, when an epistemic state appears in a propositional formula, it is always a shorthand of its belief set. For example, we write $\Psi \models \mu$ for $Bel(\Psi) \models \mu$, $\Psi \wedge \mu$ for $Bel(\Psi) \wedge \mu$, and $\omega \models \Psi$ for $\omega \models Bel(\Psi)$. In this paper, we call an operator that satisfies the above six postulates an *AGM operator*. We assume the new evidence is always satisfiable, as the revision process becomes trivial when the new evidence is unsatisfiable.

In the framework of Darwiche and Pearl, a revision operator is a function mapping an epistemic state and a proposition to a new epistemic state. Following (Katsuno and Mendelzon 1991), Darwiche and Pearl give a characterisation of epistemic states in terms of total preorders on W.

Definition 2. ((Darwiche and Pearl 1997)) A function that maps each epistemic state Ψ to a total pre-order $\preceq_\Psi$ on W is called a *faithful assignment* if and only if

(1) If $\omega_1 \models \Psi$ and $\omega_2 \models \Psi$, then $\omega_1 \sim_\Psi \omega_2$;

(2) If $\omega_1 \models \Psi$ and $\omega_2 \not\models \Psi$, then $\omega_1 \prec_\Psi \omega_2$; and

(3) If $\Psi_1 = \Psi_2$, then $\preceq_{\Psi_1} = \preceq_{\Psi_2}$.

Theorem 1. *((Darwiche and Pearl 1997)) A revision operator $\circ$ satisfies (R^*1)-(R^*6) precisely when there exists a faithful assignment that maps each epistemic state Ψ to a total preorder $\preceq_\Psi$ such that*

$$\mathit{Mod}(\Psi \circ \mu) = \min(\mathit{Mod}(\mu), \preceq_\Psi), \tag{1}$$

where $\min(\mathit{Mod}(\mu), \preceq_\Psi)$ denotes the set of minimum worlds in $\mathit{Mod}(\mu)$ under $\preceq_\Psi$.

Darwiche and Pearl establish a 1-1 correspondence between epistemic states and total preorders on worlds. On one hand, suppose Ψ is an epistemic state. For any two worlds ω_1, ω_2, by (R^*1) and (R^*3), the result of revising Ψ by $\textsc{Form}(\{\omega_1, \omega_2\})$ must be one of $\textsc{Form}(\{\omega_1\})$, $\textsc{Form}(\{\omega_2\})$, $\textsc{Form}(\{\omega_1, \omega_2\})$. From Theorem 1 and according to the different revision result, $\preceq_\Psi$ is determined as follows (Katsuno and Mendelzon 1991):

$$\begin{array}{lll} \omega_1 \prec_\Psi \omega_2 & \text{if} & \Psi \circ \textsc{Form}(\{\omega_1, \omega_2\}) \equiv \textsc{Form}(\{\omega_1\}). \\ \omega_2 \prec_\Psi \omega_1 & \text{if} & \Psi \circ \textsc{Form}(\{\omega_1, \omega_2\}) \equiv \textsc{Form}(\{\omega_2\}). \\ \omega_1 \sim_\Psi \omega_2 & \text{if} & \Psi \circ \textsc{Form}(\{\omega_1, \omega_2\}) \equiv \textsc{Form}(\{\omega_1, \omega_2\}). \end{array}$$

This gives the way to construct a total preorder from an epistemic state.

On the other hand, suppose $\preceq$ is a total preorder on W. We construct an epistemic state Ψ as follows:

(D1) $Bel(\Psi) \equiv \textsc{Form}(\min(W, \preceq))$.

(D2) A conditional belief $(\beta \mid \alpha)$ is in Ψ iff there is a world of $\alpha \wedge \beta$ which is strictly less than (by means of $\prec$) all worlds of $\alpha \wedge \neg\beta$.

(D1) follows from Thm. 1 if we let $\mu = T$ and (D2) follows from Thm. 1 and (EP) (cf. (Darwiche and Pearl 1997, Lemma 10)). Note that (D2) is equivalent to saying that every minimal world of $\alpha \wedge \beta$ (under $\preceq_\Psi$) is strictly less than every minimal world of $\alpha \wedge \neg\beta$.

We next summarise several properties of epistemic states.

Proposition 1. *Suppose Ψ is an epistemic state with $Bel(\Psi) = K$. Then we have:*

(E1) $\phi \in Bel(\Psi)$ iff $(\phi \mid T) \in \Psi$.

(E2) If $(\beta \mid \alpha) \in \Psi$, then $(\neg\beta \mid \alpha) \notin \Psi$.

(E3) If $(\beta \mid \alpha\vee\beta) \in \Psi$ and $\beta \models \gamma$, then $(\gamma \mid \alpha\vee\gamma) \in \Psi$.

(E4) If $(\beta \mid \alpha \vee \beta) \in \Psi$ and $\gamma \models \alpha$, then $(\beta \mid \gamma \vee \beta) \in \Psi$.

(E5) $(\beta \mid \alpha) \in \Psi$ and $(\gamma \mid \alpha) \in \Psi$ iff $(\beta \wedge \gamma \mid \alpha) \in \Psi$.

(E6) If $\alpha_1 \equiv \alpha_2$ and $\beta_1 \equiv \beta_2$, then $(\beta_1 \mid \alpha_1) \in \Psi$ iff $(\beta_2 \mid \alpha_2) \in \Psi$.

(E7) If ϕ is consistent, then $(\phi \mid \phi) \in \Psi$.

(E8) Suppose α, β, γ are pairwise inconsistent propositions. If $(\alpha \vee \beta \mid \alpha \vee \beta \vee \gamma) \in \Psi$, then at least one of $(\alpha \mid \alpha \vee \gamma), (\beta \mid \beta \vee \gamma)$ is in Ψ.

(E9) Suppose α, β, γ are pairwise inconsistent propositions. If $(\alpha \mid \alpha \vee \beta) \notin \Psi$ and $(\beta \mid \alpha \vee \beta) \notin \Psi$, then $(\alpha \mid \alpha \vee \gamma) \in \Psi$ iff $(\beta \mid \beta \vee \gamma) \in \Psi$.

Proof. Take (E8) as an example, the others are similar or simpler. Suppose α, β, γ are pairwise inconsistent propositions. If $(\alpha \vee \beta \mid \alpha \vee \beta \vee \gamma) \in \Psi$, then by (D2) there is a world $\omega \in \mathsf{Mod}(\alpha\vee\beta)$ such that ω is strictly less than every world of γ. Since ω is a world of either α or β, at least one of $(\alpha \mid \alpha \vee \gamma), (\beta \mid \beta \vee \gamma)$ is in Ψ. This proves (E8). □

Remark 1. *Let $\alpha = \neg\beta$ in (E3). Then (E3) is equivalent to saying the agent will believe γ if she believes β and $\beta \models \gamma$.*

We note that the first seven rules are very natural. (E2) shows that a rational agent's belief information should be consistent; (E3) and (E4) show that belief information is closed under implication; (E5) shows that belief information is closed under conjunction; and (E6) shows that epistemic state contains all equivalent belief information.

Compared to (E1)-(E7), (E8) and (E9) are perhaps too strong. It is very likely that an agent believes $(\alpha \vee \beta \mid \alpha\vee\beta\vee\gamma) \in \Psi$ but believes neither $(\alpha \mid \alpha\vee\gamma)$ nor $(\beta \mid \beta\vee\gamma)$. For example, let α (β, γ, respectively) denote the proposition that Argentina (Brazil, Germany, respectively) will win the 2014 Football World Cup. Then it is rational for an agent to believe on July 8 that the chances of Argentina and Brazil *together* is greater than that of Germany alone, but does not believe either will have a better chance than Germany.

As for (E9), even though an agent does not have either $(\alpha \mid \alpha \vee \beta)$ or $(\beta \mid \alpha \vee \beta)$ in Ψ, she may still hold a conditional belief $(\alpha \mid \alpha \vee \gamma)$ but not $(\beta \mid \alpha \vee \gamma)$, or vice versa. For example, on July 8, the agent may have no idea who will win when she believes either Argentina or Brazil will win the cup, but, still, she may believe that Argentina has a better chance than Germany once she believes either Argentina or Germany will win the cup and that Germany has a better chance than Brazil once she believes that either Brazil or Germany will win the cup.

It then seems natural to eliminate (E8) and (E9) from the definition of epistemic states.

Definition 3. Suppose Ψ consists of a belief set $Bel(\Psi)$ and a set of conditional beliefs. We call Ψ a *general epistemic state* (GEP for short) if it satisfies (E1)-(E7).

Later we will show in Thm. 6 that a GEP is an epistemic state if it satisfies (E8) and (E9). That is, the nine conditions (E1)-(E9) are necessary and sufficient for characterising epistemic states.

It is difficult (if not impossible) to represent a GEP Ψ by a total preorder on worlds without introducing new conditional beliefs which are not in Ψ (cf. Example 1). In the next section, we provide a semantical characterisation for GEPs by introducing a mathematical structure called *belief algebra*, which is in essence a binary relation defined on 2^W.

Belief algebras

Each GEP induces a natural binary relation $\gg_\Psi$ on 2^W.

Definition 4. Suppose Ψ is a GEP. We define a binary relation $\gg_\Psi$ on 2^W as follows:

- If $\phi \in Bel(\Psi)$, then $\mathsf{Mod}(\phi) \gg_\Psi \mathsf{Mod}(\neg\phi)$.
- If $(\beta \mid \alpha) \in \Psi$, then $\mathsf{Mod}(\alpha \wedge \beta) \gg_\Psi \mathsf{Mod}(\alpha \wedge \neg\beta)$.

We write $Alg(\Psi)$ for $(2^W, \gg_\Psi)$.

Proposition 2. *Suppose $U \gg_\Psi V$. Then $U \cap V = \varnothing$ and $U \gg_\Psi V$ iff $(\textsc{Form}(U) \mid \textsc{Form}(U) \vee \textsc{Form}(V)) \in \Psi$.*

Intuitively, we regard $\mathsf{Mod}(\alpha \wedge \beta) \gg \mathsf{Mod}(\alpha \wedge \neg\beta)$ as representing that $\alpha \wedge \beta$ is more believable than $\alpha \wedge \neg\beta$.

In this way, we represent the agent's belief state by a binary relation on 2^W. Note that $\mathsf{Mod}(\phi) \cap \mathsf{Mod}(\neg\phi)$ and $\mathsf{Mod}(\alpha \wedge \beta) \cap \mathsf{Mod}(\alpha \wedge \neg\beta)$ are both empty. The binary relation $\gg_\Psi$ is defined only for disjoint subsets of W. Write

$$R_W = \{(U, V) \mid U, V \subseteq W, U \cap V = \varnothing\}. \qquad (2)$$

That is, R_W consists of pairs of disjoint subsets of W. It is clear that each $\gg_\Psi$ is contained in R_W.

$(W, \gg_\Psi)$ is a belief algebra in the following sense.

Definition 5. Suppose $\gg$ is a binary relation on 2^W. We say $(2^W, \gg)$ is a *belief algebra* if it satisfies:

(A0) $\gg \subseteq R_W$.

(A1) If $U \subseteq W$, then $U \gg \varnothing$ iff $U \neq \varnothing$.

(A2) If $U \gg V$, then $V \not\gg U$.

(A3) If $U_1 \supseteq U \gg V \supseteq V_1$ and $U_1 \cap V_1 = \varnothing$, then $U_1 \gg V_1$.

(A4) If $U = U_1 \cup V_1 = U_2 \cup V_2$ and $U_1 \gg V_1, U_2 \gg V_2$, then $U_1 \cap U_2 \gg V_1 \cup V_2$.

Remark 2. *(A1) shows that each satisfiable ϕ is more believable than false; (A2) shows that if α is more believable than β then β must not be more believable than α; (A3) shows that $\gg$ satisfies conditional transitivity. (A4) is a generalisation of the following phenomenon: if α is more believable than $\neg\alpha$ and β is more believable than $\neg\beta$ then $\alpha \wedge \beta$ is more believable than $\neg\alpha \vee \neg\beta$.*

Now we show each GEP derives a belief algebra.

Theorem 2. *$Alg(\Psi)$ is a belief algebra if Ψ is a GEP.*

A belief algebra also gives rise to a GEP.

Definition 6. Suppose $G = (2^W, \gg)$ is a belief algebra. We define $Gep(G)$ which is a set of beliefs and conditional beliefs as follows:

- If $\mathsf{Mod}(\phi) \gg \mathsf{Mod}(\neg\phi)$ then $\phi \in Gep(G)$.
- If $\mathsf{Mod}(\alpha \wedge \beta) \gg \mathsf{Mod}(\alpha \wedge \neg\beta)$ then $(\beta \mid \alpha) \in Gep(G)$.

Proposition 3. *Suppose $U \cap V = \varnothing$. Then we have $(\textsc{Form}(U) \mid \textsc{Form}(U) \vee \textsc{Form}(V)) \in Gep(G)$ iff $U \gg V$.*

We then have the following result.

Theorem 3. *If $G = (2^W, \gg)$ is a belief algebra, then $Gep(G)$ is a GEP.*

Theorems 2 and 3 establish the correspondence between GEPs and belief algebras. From the definition of $Alg(\Psi)$, we know $Alg(\Psi_1) \neq Alg(\Psi_2)$ if $\Psi_1 \neq \Psi_2$. We conclude that the correspondence is 1-1. Moreover we have:

Proposition 4. *Suppose Ψ is a GEP and G is a belief algebra. Then we have*

$$Gep(Alg(\Psi)) = \Psi \text{ and } Alg(Gep(G)) = G. \tag{3}$$

Properties of belief algebra

Now we show some properties of belief algebra. The following result unveils the structure of $\gg$.

Proposition 5. *Suppose $(2^W, \gg)$ is a belief algebra. There is a unique chain $\Delta = \{U_1 \gg U_2 \gg U_3 \gg \cdots \gg U_n\}$ s.t.:*

(Ch1) $\{U_i\}_{i=1}^n$ is a partition of W.

(Ch2) For each U_i, if V_1, V_2 are two disjoint nonempty subsets of U_i, then V_1 and V_2 are incomparable in $\gg$, i.e., $(V_1, V_2) \notin \gg$ and $(V_2, V_1) \notin \gg$.

Proof. We only show how to construct Δ. If $U \gg \overline{U}$ and $V \gg \overline{V}$ then we have $U \cap V \gg \overline{U \cap V}$ by (A4) and thus $U \cap V \neq \varnothing$ by (A1). Let $U_1 = \bigcap\{U \mid U \subseteq W, U \gg \overline{U}\}$. Then $U_1 \gg \overline{U_1}$. Furthermore, for all $U \gg \overline{U}$ we have $U_1 \subseteq U$. Let $W_1 = W \setminus U_1$ (i.e. $W_1 = \overline{U_1}$). We construct U_2 in W_1 similarly as U_1 in W. Let $U_2 = \bigcap\{U \mid U \subseteq W_1, U \gg W_1 \setminus U\}$. Since $U_2 \subseteq W_1$, we have $U_1 \gg U_2$ by (A3).

Let $W_2 = W_1 \setminus U_2$. We construct a U_3 in the same way and $U_1 \gg U_2 \gg U_3$. Similarly, we get $U_4, U_5, \cdots$. Since W is a finite set, the process will stop in n steps for some $n \geq 1$. Then we hold a chain Δ. □

We call this unique chain the *backbone* of $(2^W, \gg)$. Suppose G is a belief algebra with backbone $\Delta = \{U_1 \gg U_2 \gg U_3 \gg \cdots \gg U_n\}$. From the proof of Prop. 5, we conclude that $U_1 \gg \overline{U_1}$ and $U_1 \subseteq U$ for any $U \subseteq W$ such that $U \gg \overline{U}$. Recall that $Gen(G)$ is the corresponding GEP of G. By Definition 6, U_1 consists exactly of the worlds of $Bel(Gep(G))$, i.e. $\mathsf{Mod}(Bel(Gep(G))) = U_1$.

Sparse belief algebra and complete belief algebra

In above we have seen each belief algebra has a backbone. For convenience, we also call any chain $U_1 \gg U_2 \gg U_3 \gg \cdots \gg U_n$ in 2^W that satisfies (Ch1) and (Ch2) a *backbone*.

We next show that, for every backbone Δ, there are belief algebras that have backbone Δ. Among these belief algebras, there is a smallest one and a largest one. Furthermore, we give characterisations for these special belief algebras.

Firstly, we show how to generate a belief algebra by using a subset of R_W.

Definition 7. Given $\Omega \subseteq R_W$, we denote by $Gen(\Omega)$ the smallest subset of R_W which contains Ω and is closed under (A1), (A3) and (A4).

It is possible that $Gen(\Omega)$ does not satisfy (A2). However, if $Gen(\Omega)$ satisfies (A2) then $Gen(\Omega)$ is a belief algebra. In particular, if Δ is a backbone, then $Gen(\Delta)$ satisfies (A2).

Lemma 1. *Suppose $G = (2^W, \gg)$ is a belief algebra and $\Delta = \{U_1 \gg U_2 \gg U_3 \gg \cdots \gg U_n\}$ its backbone. Let $Gen(\Delta) = (2^W, \gg^{\mathrm{sp}})$ be the belief algebra generated by Δ. Then $Gen(\Delta)$ has backbone Δ and $\gg^{\mathrm{sp}} \subseteq \gg$.*

We next examine the largest belief algebra with backbone Δ. We need the following notion of *support*.

Definition 8. Suppose $G = (2^W, \gg)$ is a belief algebra and $\Delta = \{U_1 \gg U_2 \gg U_3 \gg \cdots \gg U_n\}$ its backbone. The *support* of a nonempty set $V \subseteq W$ in G is defined as

$$I(V) = U_{\min\{i \mid V \cap U_i \neq \varnothing\}}. \tag{4}$$

By definition, $I(V)$ is the first U_j such that $V \cap U_j \neq \varnothing$.

Definition 9. Suppose $(2^W, \gg)$ is a belief algebra and $\Delta = \{U_1 \gg U_2 \gg U_3 \gg \cdots \gg U_n\}$ its backbone. We define a new binary relation $\gg^{\mathrm{c}} \subseteq R_W$ on 2^W as:

$$\gg^{\mathrm{c}} = \{(U, V) \in R_W : \ I(U) \gg I(V)\}. \tag{5}$$

Then we have

Lemma 2. *Suppose $G = (2^W, \gg)$ is a belief algebra and $\Delta = \{U_1 \gg U_2 \gg U_3 \gg \cdots \gg U_n\}$ its backbone. Then $G^{\mathrm{c}} = (2^W, \gg^{\mathrm{c}})$ is a belief algebra and has backbone Δ. Moreover, $\gg^{\mathrm{c}}$ contains $\gg$.*

From the above lemma, we conclude that, for each belief algebra G, if $U \gg V$ then $I(U) \gg I(V)$. That is to say, $I(U) \gg I(V)$ is a necessary condition for $U \gg V$. If $G = G^{\mathrm{c}}$ then $I(U) \gg I(V)$ is also a sufficient condition.

Theorem 4. *Let $G = (2^W, \gg)$ be a belief algebra with backbone $\Delta = \{U_1 \gg U_2 \gg U_3 \gg \cdots \gg U_n\}$. Among all belief algebras with backbone Δ, $Gen(\Delta)$ is the smallest and G^{c} the largest.*

We call the smallest belief algebra a *sparse* belief algebra, and the largest a *complete* belief algebra.

Definition 10. Let $G = (2^W, \gg)$ be a belief algebra with backbone $\Delta = \{U_1 \gg U_2 \gg U_3 \gg \cdots \gg U_n\}$. We call G a *sparse* belief algebra if $G = Gen(\Delta)$; and call G a *complete* belief algebra if $G = G^{\mathrm{c}}$.

The next example shows how to construct belief algebras.

Example 1. *Suppose L has two propositional atoms a and b and $W = \{\omega_1 = a \wedge b, \omega_2 = a \wedge \neg b, \omega_3 = \neg a \wedge b, \omega_4 = \neg a \wedge \neg b\}$. Let $\phi = a \wedge b$, $\alpha = \textsc{Form}(\{\omega_2, \omega_3, \omega_4\})$, $\beta = \textsc{Form}(\{\omega_2, \omega_3\})$. Assume that Bob's current belief is $\Gamma = \{\phi, (\beta \mid \alpha)\}$. That is, in the current state, Bob only believes ϕ, and he will believe β when he believes α. Suppose there is no other information about belief. From $\Gamma = \{\phi, (\beta \mid \alpha)\}$ and Definition 5 we have $\mathsf{Mod}(\phi) \gg_\Gamma \mathsf{Mod}(\neg\phi)$ and $\mathsf{Mod}(\beta \wedge \alpha) \gg_\Gamma \mathsf{Mod}(\neg\beta \wedge \alpha)$, i.e. $\gg_\Gamma$ has only two instances: $\{\omega_1\} \gg_\Gamma \{\omega_2, \omega_3, \omega_4\}$ and $\{\omega_2, \omega_3\} \gg_\Gamma \{\omega_4\}$. Let $G_1 = Gen(\gg_\Gamma)$ be the belief algebra generated by $\gg_\Gamma$ using (A0)-(A4). For the sake of brevity we use $(1, 234)$ to represent $\{\omega_1\} \gg_\Gamma \{\omega_2, \omega_3, \omega_4\}$, and, similarly, $(23, 4)$ for $\{\omega_2, \omega_3\} \gg_\Gamma \{\omega_4\}\}$. We then have*

$$G_1 = \{(1,2),(1,3),(1,4),(1,23),(1,24),(1,34),(1,234),\\ (12,3),(12,4),(12,34),(13,2),(13,4),(13,24),\\ (14,2),(14,3),(14,23),(123,4),(124,3),(134,2),\\ (23,4)\} \cup \{(U,\varnothing) \mid U \subseteq W, U \neq \varnothing\}$$

and the backbone of G_1 is $\{\omega_1\} \gg_\Gamma \{\omega_2, \omega_3\} \gg_\Gamma \{\omega_4\}$. This is, $G_1 = Gen(\{\{\omega_1\} \gg_\Gamma \{\omega_2, \omega_3\} \gg_\Gamma \{\omega_4\}\})$ is a sparse belief algebra. Adding $\{\omega_2\} \gg \{\omega_4\}$ into G_1, we get a belief algebra G_2 whose backbone is also $\{\{\omega_1\} \gg \{\omega_2, \omega_3\} \gg \{\omega_4\}\}$; Adding $\{\omega_2\} \gg \{\omega_4\}, \{\omega_3\} \gg \{\omega_4\}$ into G_1, we get a complete belief algebra G^c. It is clear that $G_1 \subset G_2 \subset G^c$.

Γ cannot be represented by a total preorder. There are several different total preorders on worlds that can entail Γ, e.g. $\omega_1 \prec \omega_2 \sim \omega_3 \prec \omega_4$ or $\omega_1 \prec \omega_2 \prec \omega_4 \prec \omega_3$ or $\omega_1 \prec \omega_3 \prec \omega_4 \prec \omega_2$. None of these preorder can represent Γ without introducing new conditional beliefs. The main reason is that the belief information determined by Γ is incomplete and is only equivalent to a GEP, viz. $Gep(G_1)$.

In the next section we establish the correspondence between epistemic states and complete belief algebras.

Epistemic states and complete belief algebras

In this section we show that if Ψ is an epistemic state then $Alg(\Psi)$ is a complete belief algebra and if G is a complete belief algebra then $Gep(G)$ is an epistemic state.

Firstly, we show that each total preorder on W leads to a complete belief algebra.

Lemma 3. *Suppose $\preceq$ is a total preorder on W. For any two nonempty subsets V_1, V_2 of W with $V_1 \cap V_2 = \varnothing$, define $V_1 \gg V_2$ if there is an $\omega_1 \in V_1$ such that $\omega_1 \prec \omega_2$ for all $\omega_2 \in V_2$. We denote by $G_\preceq = Gen(\gg)$. Then $G_\preceq$ is a complete belief algebra.*

Secondly, we show each complete belief algebra also induces a total preorder on W.

Lemma 4. *Suppose G is a complete belief algebra with backbone $\Delta = \{U_1 \gg U_2 \gg U_3 \gg \cdots \gg U_n\}$. We define a preorder on W as follows: for any $\omega_1, \omega_2 \in W$*

- *$\omega_1 \prec \omega_2$ iff $I(\{\omega_1\}) \gg I(\{\omega_2\})$.*
- *$\omega_1 \sim \omega_2$ iff $I(\{\omega_1\}) = I(\{\omega_2\})$.*

Then $\preceq$ is a total preorder and $G_\preceq = G$.

By Lemmas 3 and 4, there is a 1-1 correspondence between total preorders on W and complete belief algebras on 2^W. Recall the 1-1 correspondence between total preorders on W and epistemic states (cf. Section II). We have

Theorem 5. *Suppose Ψ is a GEP. Then Ψ is an epistemic state iff $Gen(\gg_\Psi)$ is a complete belief algebra.*

We have established the correspondence between GEPs and belief algebras in Thm. 2 and Thm. 3. This shows that the rules (A0)-(A4) for belief algebras corresponding to rules (E1)-(E7) for GEPs. Furthermore, we have the following translation of (E8) and (E9) in belief algebras:

(A5) Suppose U_1, U_2, U_3 are pairwise disjoint nonempty subsets of W. If $U_1 \cup U_2 \gg U_3$ then either $U_1 \gg U_3$ or $U_2 \gg U_3$.

(A6) Suppose U_1, U_2, U_3 are pairwise disjoint nonempty subsets of W. If $U_1 \not\gg U_2$ and $U_2 \not\gg U_1$ then $U_1 \gg U_3$ iff $U_2 \gg U_3$.

It is easy to see the following lemma.

Lemma 5. *Suppose Ψ is GEP. Then:*

(1) Ψ satisfies (E8) iff $Alg(\Psi)$ satisfies (A5).

(2) Ψ satisfies (E9) iff $Alg(\Psi)$ satisfies (A6).

From Thm. 5 and Prop. 1, it is clear that a complete belief algebra satisfies (A5)-(A6). The following result shows that any belief algebra that satisfies (A5) and (A6) is complete.

Lemma 6. *Suppose G is a belief algebra with backbone $\Delta = \{U_1 \gg U_2 \gg U_3 \gg \cdots \gg U_n\}$. If G satisfies (A5)-(A6) then G is a complete belief algebra.*

From above two lemmas, we have the following theorem.

Theorem 6. *Every GEP that satisfies (E8) and (E9) is an epistemic state.*

As a consequence of Thm. 6 and Prop. 1, we have

Theorem 7. *Suppose Ψ consists of a belief set K and a set of conditional beliefs. Then Ψ is an epistemic state iff it satisfies (E1)-(E9).*

Revising belief algebra by belief algebra

In the framework of Darwiche and Pearl (1997), an agent's current belief is represented as an epistemic state Ψ. When she gets a new evidence μ, she updates her belief to a new epistemic state $\Psi \circ \mu$. Since each epistemic state can be represented by a total preorder on the set of worlds, Darwiche and Pearl (Darwiche and Pearl 1997) essentially provide a theory for revising a total preorder by a proposition with the result being a new total preorder. There are two principles of their framework. One is the *success* principle: μ should be put into $Bel(\Psi \circ \mu)$; the other one is the *minimal change* principle, which requires the agent to keep the ordering in $\preceq_\Psi$ to $\preceq_{\Psi\circ\mu}$ as much as possible. The relation between $\preceq_\Psi$ and $\preceq_{\Psi\circ\mu}$ is constrained as follows:

(CR1) If $\omega_1, \omega_2 \in \mathsf{Mod}(\mu)$ then $\omega_1 \preceq_{\Psi\circ\mu} \omega_2$ iff $\omega_1 \preceq_\Psi \omega_2$.

(CR2) If $\omega_1, \omega_2 \in \mathsf{Mod}(\neg\mu)$ then $\omega_1 \preceq_{\Psi\circ\mu} \omega_2$ iff $\omega_1 \preceq_\Psi \omega_2$.

(CR3) If $\omega_1 \in \mathsf{Mod}(\mu)$ and $\omega_2 \in \mathsf{Mod}(\neg\mu)$, then $\omega_1 \prec_\Psi \omega_2$ implies $\omega_1 \prec_{\Psi\circ\mu} \omega_2$.

(CR4) If $\omega_1 \in \mathsf{Mod}(\mu)$ and $\omega_2 \in \mathsf{Mod}(\neg\mu)$, then $\omega_1 \preceq_\Psi \omega_2$ implies $\omega_1 \preceq_{\Psi\circ\mu} \omega_2$.

Recall by Lemma 3 we know $\preceq_\Psi$ and $\preceq_{\Psi\circ\mu}$ can be represented as complete belief algebras. A proposition μ can also be represented as the sparse belief algebra $Alg(\mu)$ generated by $\mathsf{Mod}(\mu) \gg \mathsf{Mod}(\neg\mu)$. Thus we regard the revision framework of Darwiche and Pearl as revising a complete belief algebra by a sparse belief algebra $Alg(\mu)$.

In this section, we consider a more general revision framework. We suppose the agent's current belief G_1, the new evidence G_2, and the result of the revision G_3 are all belief algebras. The process is represented by an operator as $G_1 \bullet G_2 = G_3$.

Suppose $A_1 \gg_1 A_2 \gg_1 \cdots \gg_1 A_n$ and $B_1 \gg_2 B_2 \gg_2 \cdots \gg_2 B_m$ are the backbones of G_1 and, respectively, G_2. Firstly, from the success principle, all the ordering of G_2 should be put into G_3. That is,

(RA1) If $U \gg_2 V$ then $U \gg_3 V$.

Secondly, we extend (CR1)-(CR4) to the general framework. (CR1) and (CR2) show that $\preceq_\Psi$ and $\preceq_{\Psi\circ\mu}$ are identical in $\mathsf{Mod}(\mu)$ and $\mathsf{Mod}(\neg\mu)$. Note that $\mathsf{Mod}(\mu)$ and $\mathsf{Mod}(\neg\mu)$ are the only elements of the backbone of $Alg(\mu)$. Suppose B_i is an element in G_2's backbone. We require $\gg_1$ and $\gg_3$ to be identical in B_i.

(RA2) If $U, V \subseteq B_i$, then $U \gg_3 V$ iff $U \gg_1 V$;

Now consider how to extend (CR3). Suppose $\omega_1 \in \mathsf{Mod}(\mu)$ and $\omega_2 \in \mathsf{Mod}(\neg\mu)$. Note that $I(\{\omega_1\}) = \mathsf{Mod}(\mu)$ and $I(\{\omega_2\}) = \mathsf{Mod}(\neg\mu)$ in $Alg(\mu)$. (CR3) can be rephrased as "If $\{\omega_1\} \gg_\Psi \{\omega_2\}$ and $I_\mu(\{\omega_1\}) \gg_\mu I_\mu(\{\omega_2\})$ in $Alg(\mu)$, then $\{\omega_1\} \gg_{\Psi\circ\mu} \{\omega_2\}$." More generally, we have

(RA3) If $U \gg_1 V$, $I(U) \gg_2 I(V)$ in G_2, then $U \gg_3 V$.

From (RA1) and (RA2) we have already put the whole $\gg_2$ and part of $\gg_1$ into $\gg_3$. To make sure (RA3) does not incur inconsistency, we need to prove that (V, U) is not in $\gg_3$. Because $U \gg_1 V$ and $I(U) \gg_2 I(V)$ in G_2, (V, U) is in neither $\gg_1$ nor $\gg_2$. This guarantees (V, U) is not in $\gg_3$.

Since $\omega_1 \preceq_\Psi \omega_2$ is equivalent to $\omega_2 \not\prec \omega_1$, (CR4) can be rephrased as "If $(\{\omega_2\}, \{\omega_1\}) \notin \gg_\Psi$ and $I_\mu(\{\omega_1\}) \gg_\mu I_\mu(\{\omega_2\})$, then $(\{\omega_2\}, \{\omega_1\}) \notin \gg_{\Psi\circ\mu}$." Similarly, we have

(RA4) If $(U, V) \notin \gg_1$, $I(V) \gg_2 I(U)$ in G_2, then $(U, V) \notin \gg_3$.

To summarise, we have presented a framework that revises belief algebra by belief algebra. Our revision rules (RA1)-(RA4) are natural extensions of the rules in (Darwiche and Pearl 1997).

We define a concrete operator that satisfies (RA1)-(RA4).

Proposition 6. *Suppose G_1, G_2 are belief algebras with backbones $A_1 \gg_1 A_2 \gg_1 \cdots \gg_1 A_n$ and $B_1 \gg_2 B_2 \gg_2 \cdots \gg_2 B_m$ respectively. Let $G_1 \star G_2 = Gen(\gg)$, where $\gg$ is defined as follows:*

(S1) If $U \gg_2 V$, then $U \gg V$;

(S2) If $U, V \subseteq B_i$ and $U \gg_1 V$, then $U \gg V$;

(S3) If $U \gg_1 V$ and $I(U) \gg_2 I(V)$ in G_2, then $U \gg V$.

Then $G_1 \star G_2$ is a belief algebra and $\star$ satisfies (RA1)-(RA4). Moreover, for any $\bullet$ satisfying (RA1)-(RA4), we have $G_1 \star G_2 \subseteq G_1 \bullet G_2$.

The above proposition shows $\star$ is the most conservative revision operator. The revision result of $G_1 \star G_2$ is generated by all necessary belief information which should be kept. It is worth noticing that these necessary belief information may be not sufficient to induce a total preorder. However, it can always generate a belief algebra, viz. $Gen(\gg)$.

We show an application of $\star$ in the next example.

Example 2. *In Example 1, Bob's current belief algebra is $G_1 = Gen(\{\{\omega_1\} \gg_1 \{\omega_2, \omega_3, \omega_4\}, \{\omega_2, \omega_3\} \gg_1 \{\omega_4\}\})$ with backbone $\{\omega_1\} \gg_1 \{\omega_2, \omega_3\} \gg_1 \{\omega_4\}$. Suppose he learns an evidence $\psi = \textsc{Form}(\omega_2, \omega_4)$ or, but equivalently, a belief algebra $Alg(\psi)$ with backbone $\{\omega_2, \omega_4\} \gg_2 \{\omega_1, \omega_3\}$. From $\{\omega_1\} \gg_1 \{\omega_3\}$, $\{\omega_2\} \not\gg_1 \{\omega_4\}$ and $\{\omega_4\} \not\gg_1 \{\omega_2\}$, it is easy to check $G_1 \star Alg(\psi) = Gen(\{\omega_2, \omega_4\} \gg \{\omega_1\} \gg \{\omega_3\})$ and $\{\omega_2, \omega_4\} \gg \{\omega_1\} \gg \{\omega_3\}$ is the backbone of $G_1 \star Alg(\psi)$. Since the belief set is decided by the largest elements in backbone, we have $Bel(G_1 \star Alg(\psi)) \equiv \textsc{Form}(\{\omega_2, \omega_4\})$.*

Related work

The epistemic state approach to belief revision proposed in (Darwiche and Pearl 1997) is one of the most influential approaches for iterated belief revision. There are some other earlier classical works (e.g. (Gärdenfors 1988; Nebel 1991)) which are related to epistemic states and iterated belief revision. In particular, Darwiche and Pearl pointed out that their work was partially inspired by studies of (Freund and Lehmann 1994) and the absolute minimisation principle of changes in conditional beliefs was due to (Boutilier 1996).

In this paper we use a binary relation $\gg$ on 2^W to characterise belief. Recall that, for each $U \in 2^W$, there is a proposition $\textsc{Form}(U)$ in L such that $\mathsf{Mod}(\textsc{Form}(U)) = U$. It is then natural to extend $\gg$ to a binary relation on L. Gardenfors and Makinson (Gärdenfors and Makinson 1988) introduce a total preorder on propositions, known as *epistemic entrenchment*. Roughly speaking, an epistemic entrenchment related to a theory K is defined as a total preorder $\leq$ on L for characterising the relative importance of sentences when belief changes. It is not difficult to establish a 1-1 correspondence between epistemic entrenchments and epistemic states. By results obtained in this paper, we can see epistemic entrenchments also correspond to complete belief algebras.

Several researchers have already observed that total preorders on worlds are not suitable for representing some types of belief information. For example, (Katsuno and Mendelzon 1991; Benferhat, Lagrue, and Papini 2005) propose to use *partial* preorders on worlds for representing belief information; and (Peppas and Williams 2014) use *semiorders* on *worlds* to characterise some other types of belief information. All these work are, however, based on orders on worlds. Similarly as representing a total preorder by a belief algebra, we can show partial preorders or semiorders on worlds are special cases of belief algebras, but there are belief algebras which can not be precisely characterised by (partial or semi) orders on worlds. This shows that our belief algebra model is more general for representing belief information.

Another way to represent belief information is using probabilistic measures on set of worlds. In this case, belief revision is revising a probabilistic measure to a new probabilistic measure by new evidence, which satisfies Jeffrey's rules (Ma, Liu, and Benferhat 2010; Benferhat et al. 2010).

Our belief algebra also has a close relation to qualitative plausibility measure, which is used in default reasoning (Friedman and Halpern 2001). In fact, we can construct a qualitative plausibility measure in a natural way from the backbone of a belief algebra. More connections will be investigated in our future work.

Conclusion

This paper provided a generalised framework of belief revision. We first gave an axiomatic characterisation of the concept of epistemic state. Suppose Ψ consists of a belief set and conditional beliefs. We showed that Ψ can be characterised by a total preorder on worlds iff Ψ satisfies (E1)-(E9); and argued that (E8), (E9) are too strong for characterising belief information. A structure which satisfies (E1)-(E7) is called a general epistemic state (GEP). We also provided a semantical characterisation for GEPs in terms of belief algebras. Using the notion of belief algebras, we extended the classical belief revision rules of Darwiche and Pearl to our general setting of GEPs. Belief revision is then characterised as revising a belief algebra by a belief algebra. Our belief algebra model is more general than other models based on (partial or semi) orders on worlds. In particular, total preorders on worlds correspond to complete belief algebras.

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References

Alchourron, C. E.; Gärdenfors, P.; and Makinson, D. 1985. On the logic of theory change: Partial meet contraction and revision functions. *J. Symb. Log.* 50(2):510–530.

Benferhat, S.; Dubois, D.; Prade, H.; and Williams, M.-A. 2010. A framework for iterated belief revision using possibilistic counterparts to Jeffrey's rule. *Fundam. Inform.* 99(2):147–168.

Benferhat, S.; Lagrue, S.; and Papini, O. 2005. Revision of partially ordered information: Axiomatization, semantics and iteration. In *IJCAI*, 376–381.

Boutilier, C. 1996. Iterated revision and minimal change of conditional beliefs. *Journal of Philosophical Logic* 25(3):263–305.

Darwiche, A., and Pearl, J. 1997. On the logic of iterated belief revision. *Artificial intelligence* 89(1):1–29.

Delgrande, J. P. 2012. Revising beliefs on the basis of evidence. *International Journal of Approximate Reasoning* 53(3):396–412.

Freund, M., and Lehmann, D. 1994. Belief revision and rational inference. Technical Report TR-94-16, The Hebrew University of Jerusalem, Jerusalem, Institute of Computer Science.

Friedman, N., and Halpern, J. Y. 2001. Plausibility measures and default reasoning. *Journal of the ACM* 48(4):648–685.

Gärdenfors, P., and Makinson, D. 1988. Revisions of knowledge systems using epistemic entrenchment. In *Proceedings of the 2nd Conference on Theoretical Aspects of Reasoning about Knowledge*, 83–95.

Gärdenfors, P. 1988. *Knowledge in flux: Modeling the dynamics of epistemic states*. The MIT press.

Jin, Y., and Thielscher, M. 2007. Iterated belief revision, revised. *Artificial Intelligence* 171(1):1–18.

Katsuno, H., and Mendelzon, A. O. 1991. Propositional knowledge base revision and minimal change. *Artificial Intelligence* 52(3):263–294.

Kern-Isberner, G. 1999. Postulates for conditional belief revision. In *IJCAI*, 186–191.

Ma, J.; Benferhat, S.; and Liu, W. 2012. Revising partial preorders with partial pre-orders: A unit-based revision framework. In *KR*.

Ma, J.; Liu, W.; and Benferhat, S. 2010. A belief revision framework for revising epistemic states with partial epistemic states. In *AAAI*, 333–338.

Nebel, B. 1991. Belief revision and default reasoning: Syntax-based approaches. In *Proceedings of the 2nd International Conference on Principles of Knowledge Representation and Reasoning, KR 1991*, 417–428.

Peppas, P., and Williams, M.-A. 2014. Belief change and semiorders. In *Principles of Knowledge Representation and Reasoning: Proceedings of the Fourteenth International Conference, KR 2014*.

Spohn, W. 1988. Ordinal conditional functions: A dynamic theory of epistemic states. Springer.

Tamargo, L. H.; Falappa, M. A.; García, A. J.; and Simari, G. R. 2011. A change model for credibility partial order. In *Scalable Uncertainty Management*. Springer. 317–330.