

Cooperative Games with Bounded Dependency Degree

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Abstract

Cooperative games provide a framework to study cooperation among self-interested agents. They offer a number of solution concepts describing how the outcome of the cooperation should be shared among the players. Unfortunately, computational problems associated with many of these solution concepts tend to be intractable—NP-hard or worse. In this paper, we incorporate complexity measures recently proposed by Feige and Izsak (2013), called *dependency degree* and *supermodular degree*, into the complexity analysis of cooperative games. We show that many computational problems for cooperative games become tractable for games whose dependency degree or supermodular degree are bounded. In particular, we prove that simple games admit efficient algorithms for various solution concepts when the supermodular degree is small; further, we show that computing the Shapley value is always in FPT with respect to the dependency degree. Finally, we observe that, while determining the dependency among players is computationally hard, there are efficient algorithms for special classes of games.

1 Introduction

Cooperative games provide a convenient framework to study cooperation among self-interested agents. Formally, a *cooperative transferable utility game*, or simply a *game*, is a pair (N, v) where $N = \{1, 2, \dots, n\}$ is a finite set of *players* and $v : 2^N \rightarrow \mathbb{R}$ is a *characteristic function*. We are interested in how players should divide the value $v(N)$ of the *grand coalition* N . To capture the idea of a stable or fair payoff division scheme, a number of solution concepts have been developed, such as the core and the Shapley value. Unfortunately, the computational problems associated with many of these solution concepts are often intractable—NP-hard or worse.

There are two ways to circumvent computational intractability in this context. The first approach is to identify interesting classes of games that admit efficient algorithms. For instance, it is well-known that when the characteristic function v is convex, an outcome in the core can be computed by a polynomial-time algorithm. However, this algorithm offers no guarantees when the input game is not convex, even if the convexity constraint is only violated at

a few points, and hence this approach is of limited value. A more flexible way to cope with hardness is to provide complexity guarantees for all instances so that the guarantee depends on the complexity of the instance, that is, to design algorithms whose running time depends on how well the input instance is structured. In the context of cooperative games, this approach has been pursued by Ieong and Shoham (2005), who propose a representation formalism, which they call *marginal contribution nets (MC-nets)*, and design an algorithm for computing an allocation in the core whose running time depends on the treewidth of the graph associated with the MC-net representation of the input game.

In this work, we explore the power of the latter approach for two measures of structural complexity of set functions that have been recently developed by Feige and Izsak (2013): the *dependency degree* and the *supermodular degree*. Intuitively, the complexity of a set function v is measured using the notion of *dependency* among players at v ; such dependencies induce a graph describing the relation between players—the *dependency graph*. A player’s dependency degree is her degree in this graph; her supermodular degree is her degree in a modified version of this graph, which only takes certain dependencies into account.

Our contribution. We argue that both the dependency degree and the supermodular degree are useful in the context of cooperative games, both analytically and computationally.

We show that several cooperative game theory concepts can be naturally interpreted in terms of a dependency graph. In particular, a player’s dependency degree reflects on her importance in the game: in a simple game, a dummy player is an isolated vertex of the dependency graph and a veto player is a vertex with the maximum degree. We can also relate properties of a game, such as its dimension, to the properties of its dependency graph: for instance, we show that dependency graphs of weighted voting games are clique-trees, namely, chordal graphs.

We then investigate which solution concepts in cooperative games can be computed efficiently if the dependency/supermodular degree is bounded. For simple games, we obtain a number of tractability results with respect to the supermodular degree. Specifically, we prove that simple games admit an efficient algorithm for computing an element of the core or the least core when the supermodular degree is small. Further, while finding an optimal coalition

structure is computationally intractable even for weighted voting games, we show that this problem becomes tractable for weighted voting games with small supermodular degree. However, these results do not extend to general games: we prove that the associated separation problem for the least core is NP-hard even for games with constant dependency degree. On the other hand, we show that computing the Shapley value and the Banzhaf value is in FPT with respect to the dependency degree.

We also consider the problem of computing the dependency degree and the supermodular degree given various representations of a game. While intractability turns out to be inevitable in general, we provide polynomial and pseudo-polynomial algorithms for special classes of games.

Related work. Computational aspects of cooperative games have received a considerable amount of attention over the last few decades; we refer the reader to the book of Chalkiadakis, Elkind, and Wooldridge (2011).

Our work is similar in spirit to the complexity study of induced subgraph games or, more broadly, games defined by MC-nets (Deng and Papadimitriou 1994; Jeong and Shoham 2005; Greco et al. 2011; 2014; Li and Conitzer 2014). Indeed, each MC-net induces an *agent graph*, which also aims to capture dependencies among agents. However agent graphs are defined in a purely syntactic manner, by looking at agents' co-occurrences in the rules of an MC-net, whereas the dependency graph is defined semantically, i.e., in terms of the value of the characteristic function. Moreover, both the dependency degree and the supermodular degree are different from the concepts that are usually used to measure the complexity of an agent graph (such as treewidth).

There are also similarities between our model and *Myerson games* (Myerson 1977; Chalkiadakis, Greco, and Markakis 2016; Meir et al. 2013; Igarashi 2017) where players are located on a graph and coalitions are only allowed to form if they are connected in this graph. However, in Myerson games non-adjacent agents may still depend on each other, and hence an agent's dependency degree may be high even if her degree in the underlying Myerson graph is small. As a consequence, some problems that are easy for games with small supermodular degree remain hard for games on bounded-degree graphs, even if these graphs are acyclic (Igarashi 2017). Our results for coalition structure generation (Section 6) are similar in spirit to those of Voice, Polukarov, and Jennings (2012); we discuss the relationship between their results and ours in Section 6.

The dependency degree and the supermodular degree have been introduced by Feige and Izsak (2013), who showed applications of these measures to the welfare maximization problem. Feldman and Izsak (2014) generalized these results to function maximization subject to k -extendible system constraints (a generalization of the intersection of k matroids). These concepts have also been applied in an online setting (Feldman and Izsak 2017), and in the context of efficiency of auctions (Feldman et al. 2016), optimization of SDN upgrades (Poularakis et al. 2017) and committee selection (Izsak 2017). Some related complexity measures are the submodularity ratio (Das and Kempe 2011) and MPH (Feige et al. 2015).

2 Preliminaries

We start by defining basic notation and terminology of cooperative games. Recall that a cooperative game is a pair (N, v) , where N is a finite set and v is a function from 2^N to \mathbb{R} . Throughout the paper, we assume $v(\emptyset) = 0$. For $s \in \mathbb{N}$, let $[s] = \{1, 2, \dots, s\}$. For a vector $\mathbf{x} \in \mathbb{R}^n$ and a subset $S \subseteq N$ we use the notation $x(S) = \sum_{i \in S} x_i$, with the convention that $x(\emptyset) = 0$. The subsets of N are referred to as *coalitions*. An *imputation* for a game (N, v) is a vector $\mathbf{x} \in \mathbb{R}^N$ satisfying *efficiency*: $x(N) = v(N)$, and *individual rationality*: $x_i \geq v(\{i\})$ for every $i \in N$. For a player $i \in N$ and a coalition $S \subseteq N \setminus \{i\}$, we let $v(i|S) = v(S \cup \{i\}) - v(S)$, that is, $v(i|S)$ is the *marginal contribution* of i to S . A player i is called a *dummy* if she does not contribute to any coalition, i.e., $v(i|S) = 0$ for every $S \subseteq N$.

A set function $v : 2^N \rightarrow \mathbb{R}$ is said to be *monotone* if for every pair of subsets $S, T \subseteq N$ it holds that $S \subseteq T$ implies $v(S) \leq v(T)$. A game (N, v) is said to be *simple* if v is monotone and only takes values in $\{0, 1\}$. In a simple game, coalitions of value 1 are said to be *winning*, and coalitions of value 0 are said to be *losing*. A winning coalition S is said to be *minimal* if removal of any player from S makes it losing, i.e., $S \setminus \{i\}$ is losing for every $i \in S$. A player i is said to be a *veto player* if she is present in all winning coalitions, i.e., if a coalition S is winning, then $i \in S$. A player i is a *pivot* for a coalition $S \subseteq N \setminus \{i\}$ if S is losing and $S \cup \{i\}$ is winning.

The *core* is a classic solution concept in cooperative games. Formally, the core of a game (N, v) is the set of all imputations \mathbf{x} such that no coalition has an incentive to defect from \mathbf{x} , i.e., $x(S) \geq v(S)$, for all $S \subseteq N$. As the core can be empty, and, on the other hand, not all outcomes in the core are equally fair, we consider the *least core*, which can be thought of as the set of most stable outcomes. We first define the *excess* of a coalition $S \in 2^N \setminus \{N, \emptyset\}$ at an imputation \mathbf{x} as $e(\mathbf{x}, S) := v(S) - x(S)$; intuitively, $e(\mathbf{x}, S)$ is the degree of unhappiness of S at \mathbf{x} . The *least core* of a game (N, v) is the set of all imputations \mathbf{x} that minimize the maximum excess, i.e., the set of optimal solutions \mathbf{x} of the following linear program:

$$\begin{aligned}
 (\text{LP}_0) \quad & \min \quad \varepsilon \\
 \text{s.t.} \quad & x(S) \geq v(S) - \varepsilon \quad \text{for all } S \in 2^N \setminus \{N, \emptyset\} \\
 & x_i \geq v(\{i\}) \quad \text{for all } i \in N \\
 & x(N) = v(N).
 \end{aligned}$$

We also consider solution concepts capturing fairness among players: the Shapley value and the Banzhaf value. The *Shapley value* $\phi_i(N, v)$ of a player $i \in N$ in a game (N, v) is the average of i 's marginal contributions at v over all permutations of the players, that is,

$$\phi_i(N, v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(n - |S| - 1)!}{n!} v(i|S).$$

The *Banzhaf value* is the average of i 's marginal contribu-

	General games	Simple games	Weighted voting games
Optimal coalition structure	NP-h.	NP-h. for $p = 6$ (Th. 11)	FPT wrt p (Cor. 2)
Core		P	P
Least Core		FPT wrt p (Th. 5)	FPT wrt p
Shapley	FPT wrt d (Th. 8)	FPT wrt d	FPT wrt d
Banzhaf	FPT wrt d (Th. 9)	FPT wrt d	FPT wrt d

Table 1: Overview of complexity results for computing various solution concepts when parameterized by the dependency degree d and the supermodular degree p . Note that deciding the non-emptiness of the core for a simple game is straightforward: the core is non-empty if and only if there is a veto player (Chalkiadakis, Elkind, and Wooldridge 2011).

tions at v over all coalitions, that is,

$$\beta_i(N, v) = \frac{1}{2^{n-1}} \sum_{S \subseteq N \setminus \{i\}} v(i|S).$$

Computational setting Throughout the paper, we only consider games (N, v) such that v is computable in time polynomial in n . Furthermore, ‘polynomial’ always means polynomial in the number of players n . Note that the explicit representation of a game (N, v) , which lists the values of all coalitions, is not polynomial in n ; thus, for our computational results for general games we assume oracle access to the characteristic function v . We say that a problem is *fixed parameter tractable* (FPT) with respect to a parameter k if each instance I of this problem can be solved in time $f(k)\text{poly}(|I|)$, where f is a computable function that depends on k only.

We omit some proofs due to space constraints; the omitted proofs can be found in the full version of the paper (Igarashi, Izsak, and Elkind 2017).

3 Dependency Graphs of Cooperative Games

In this section, we introduce complexity measures representing dependencies among players in a cooperative game (Feige and Izsak 2013), and study how well such parameters capture important concepts in cooperative games.

Given a game (N, v) and two players $i, j \in N$, we say that player i *positively depends on* player j if there exists a coalition $S \subseteq N \setminus \{i, j\}$ such that $v(i|S \cup \{j\}) > v(i|S)$, i.e., i can contribute more to S in the presence of j . We say that i *depends on* j if there exists a coalition $S \subseteq N \setminus \{i, j\}$ such that $v(i|S \cup \{j\}) \neq v(i|S)$, i.e., i ’s contribution to S depends on the presence of j . These relations are known to be symmetric (Feige and Izsak 2013); hence, we can capture the dependency relations between players by undirected graphs. Formally, we define the *supermodular dependency graph* G_v^+ to be an undirected graph where the set of nodes is given by N and the set of edges is given by the pairs of players positively depending on each other. The *supermodular dependency set* of i at v is

$$D^+(i) = \{j \in N \setminus \{i\} \mid i \text{ positively depends on } j\}.$$

The *supermodular degree* p of $v : 2^N \rightarrow \mathbb{R}$ is defined as the maximum size of the supermodular dependency set, i.e.,

$p = \max_{i \in N} |D^+(i)|$. We define the *dependency graph* G_v to be an undirected graph where the set of nodes is given by N and the set of edges is given by the pairs of players depending on each other. The *dependency set* of i at v is

$$D(i) = \{j \in N \setminus \{i\} \mid i \text{ depends on } j\}.$$

The *dependency degree* d of $v : 2^N \rightarrow \mathbb{R}$ is defined as the maximum size of the dependency set, i.e., $d = \max_{i \in N} |D(i)|$.

Example 1. Consider a simple game (N, v) with player set $N = [4]$ and a characteristic function $v : 2^N \rightarrow \{0, 1\}$ where the set of minimal winning coalitions is $\{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 4\}\}$. It is easy to see that only such pairs have a positive dependence. Further, player 1 depends on player 3 since 1 can make a positive marginal contribution to the coalition $\{2, 4\}$, but her contribution becomes zero in the presence of 3. A similar argument applies to the pair 2, 4. The resulting dependency graphs G_v^+ and G_v are depicted in Figure 1.



Figure 1: A supermodular dependency graph and a dependency graph (Example 1)

We will now show that the parameters defined above capture the importance of a player in the underlying game. The theorem below shows that a dummy player corresponds to an isolated node in the dependency graph.

Theorem 1. *For every cooperative game (N, v) and every player $i \in N$, the following statements are equivalent:*

- (i) *Player i is a dummy player.*
- (ii) $v(\{i\}) = 0$ and $D(i) = \emptyset$.
- (iii) $v(\{i\}) = 0$ and $D^+(i) = \emptyset$.

Proof. (i) \implies (ii): Suppose that i is a dummy player. Then we have $v(\{i\}) = 0$. Now, suppose that there exists a player $j \in D(i)$. Then there is a coalition S such that at least one of $v(i|S)$ and $v(i|S \cup \{j\})$ is non-zero, contradicting the fact that i is a dummy player.

(ii) \implies (iii): This direction holds by the definition.
(iii) \implies (i): Suppose that $v(\{i\}) = 0$ and $D^+(i) = \emptyset$. Assume towards a contradiction that $v(i|S) \neq 0$ for some coalition $S \subseteq N \setminus \{i\}$. Let S^* be a minimal coalition with respect to this property. If $S^* \neq \emptyset$, then there is a player $j \in S^*$ such that $v(i|S^* \setminus \{j\}) = 0$, which means that i can contribute more in the presence of j , and hence i positively depends on j , a contradiction. If $S^* = \emptyset$, it follows that $v(\{i\}) \neq 0$, a contradiction again. \square

Positive Dependence in Simple Games

We will now further investigate the structure of dependency graphs in simple games. It turns out that such games can be almost fully characterized by the positive dependence relation. We first observe that in simple games the dependency relation admits a natural interpretation.

Lemma 1. *Consider a simple game (N, v) and two players $i, j \in N$. Player i positively depends on player j if and only if there exists a coalition $S \subseteq N \setminus \{i, j\}$ such that $S \cup \{i, j\}$ is winning, but the coalitions $S \cup \{i\}$ and $S \cup \{j\}$ are losing.*

Now, in contrast with dummy players, a veto player in a simple game is adjacent to every non-dummy player in the dependency graph.

Theorem 2. *Let i be a veto player in a simple game (N, v) . Then i positively depends on each non-dummy player $j \in N \setminus \{i\}$.*

Proof. Take any non-dummy player $j \in N \setminus \{i\}$. Then j is a pivot for some coalition $S \subseteq N \setminus \{j\}$, i.e., $S \cup \{j\}$ is winning and S is losing. Since i is a veto player and $i \neq j$, we have $i \in S$. Let $S_{-i} = S \setminus \{i\}$. Then, the coalition $S_{-i} \cup \{i, j\}$ is winning, and coalition $S_{-i} \cup \{i\}$ is losing. Also, as $i \notin S_{-i} \cup \{j\}$, coalition $S_{-i} \cup \{j\}$ is losing. By Lemma 1, we conclude that i positively depends on j . \square

We also observe that all minimal winning coalitions correspond to cliques in the supermodular dependency graph.

Theorem 3. *In a simple game (N, v) player $i \in N$ positively depends on a player $j \in N$ if and only if there exists a minimal winning coalition $S \subseteq N$ such that $i, j \in S$.*

Proof. Suppose that i positively depends on j . By Lemma 1 there exists a coalition $S \subseteq N \setminus \{i\}$ such that $S \cup \{i, j\}$ is winning, but coalitions $S \cup \{j\}$ and $S \cup \{i\}$ are losing. Choose a minimal coalition S^* with respect to this property. Then, removing any player from $S^* \cup \{i, j\}$ makes this coalition losing, and hence $S^* \cup \{i, j\}$ is a minimal winning coalition. Conversely, suppose that i, j belong to some minimal winning coalition S . Since S is a minimal winning coalition, both $S \setminus \{j\}$ and $S \setminus \{i\}$ are losing, but S is winning. By Lemma 1, i positively depends on j . \square

Corollary 1. *Consider a simple game (N, v) and a minimal winning coalition $S \subseteq N$. Any two distinct players $i, j \in S$ positively depend on each other.*

Weighted voting games form a subclass of simple games. Such games can be succinctly represented by the *weight* of each player, and a *quota*. Formally, a weighted voting game

with a set of players $N = [n]$ is given by a list of non-negative weights $\mathbf{w} = (w_1, w_2, \dots, w_n)$ and a quota q ; we will write $[N; \mathbf{w}; q]$. Its characteristic function $v : 2^N \rightarrow \{0, 1\}$ is given by

$$v(S) = \begin{cases} 1 & \text{if } \sum_{i \in S} w_i \geq q \\ 0 & \text{otherwise.} \end{cases}$$

It turns out that the supermodular dependency graph of a weighted voting game has a special structure: we will show that any such graph is a *chordal graph*. Recall that a graph is said to be *chordal* if any cycle of four or more nodes has a *chord*, i.e., an edge that does not belong to the cycle, but connects two of its nodes.

We first state and prove the following lemma.

Lemma 2. *Consider a weighted voting game $[N; \mathbf{w}; q]$ where a player $i \in N$ positively depends on a player $k \in N$ and $w_i \geq w_j \geq w_k$. Then, player i positively depends on player j .*

Proof. Since i positively depends on k , there is a coalition S such that $S \cup \{i, k\}$ is winning, but $S \cup \{i\}$ and $S \cup \{k\}$ are losing. Now, since $w_i \geq w_j \geq w_k$, $S \cup \{i, j\}$ is winning, but $S \cup \{i\}$ and $S \cup \{j\}$ are losing, implying that i positively depends on j . \square

Theorem 4. *For every weighted voting game $[N; \mathbf{w}; q]$ it holds that its supermodular dependency graph is chordal.*

Proof. Suppose towards a contradiction that G_v^+ has a chordless cycle C of at least four players. Let $C = \{i_1, i_2, \dots, i_k\}$, where i_j positively depends on i_{j+1} for $j \in [k]$ (with the convention that $i_{k+1} = i_1$). Assume without loss of generality that i_1 has the maximum weight among the players in C and $w_{i_2} \leq w_{i_k}$.

First, suppose that there is a player $i_j \in C \setminus \{i_1, i_2, i_k\}$ such that $w_{i_j} \geq w_{i_2}$. Since i_1 positively depends on i_2 and $w_{i_1} \geq w_{i_j} \geq w_{i_2}$, by Lemma 2 this means that i_1 positively depends on i_j , a contradiction. Now suppose that for all players $i_j \in C \setminus \{i_1, i_2, i_k\}$ we have $w_{i_j} < w_{i_2}$; in particular, we have $w_{i_{k-1}} < w_{i_2}$. Since i_k positively depends on i_{k-1} , and $w_{i_k} \geq w_{i_2} \geq w_{i_{k-1}}$, by Lemma 2 player i_k positively depends on i_2 , a contradiction. \square

A simple game (N, v) is the *intersection* of k weighted voting games $[N; \mathbf{w}^\ell; q^\ell]$, $\ell \in [k]$, if for every coalition $S \subseteq N$ we have $v(S) = 1$ if and only if $w^\ell(S) \geq q^\ell$ for all $\ell \in [k]$. It is known that every simple game G can be represented as an intersection of multiple weighted voting games; the minimum number of weighted voting games whose intersection equals to G is called the *dimension* of G .

Observe that the game defined in Example 1 has dimension 2: it can be represented as the intersection of weighted voting games $[N; \mathbf{w}^1; 1]$ and $[N; \mathbf{w}^2; 1]$, where $w^1(1) = w^1(3) = w^2(2) = w^2(4) = 1$, and all other weights are zero. However, its supermodular dependency graph has a chordless cycle of length four; thus, we cannot guarantee that the supermodular dependency graph of a simple game is chordal beyond dimension 1.

4 Complexity of Stability-Related Solution Concepts

In this section, we investigate the complexity of computing outcomes in the core and the least core.

For simple games, it is well-known that deciding if the core is not empty or finding an outcome in the core is easy. We now complement this result by showing that computing an element of the least core in simple games is fixed-parameter tractable with respect to the supermodular degree.

Theorem 5. *Let (N, v) be a simple game. Given its supermodular dependency graph G_v^+ and oracle access to v , we can compute an element of the least core in FPT time with respect to the supermodular degree.*

Proof. We first check whether the input game admits an imputation, i.e., whether $v(N) \geq \sum_{i \in N} v(\{i\})$: if not, the least core is empty. Thus, from now on we assume that $v(N) \geq \sum_{i \in N} v(\{i\})$.

It suffices to show that the separation problem for the linear program LP_0 is fixed-parameter tractable with respect to the supermodular degree. Fix an $\varepsilon \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^N$. First, we can clearly check in polynomial time whether \mathbf{x} is an imputation; thus, in the rest of the proof we assume that this is indeed the case. We need to show that deciding whether the following inequality holds is in FPT with respect to p :

$$\min\{x(S) - v(S) \mid S \in 2^N \setminus \{N, \emptyset\}\} \geq \varepsilon. \quad (1)$$

Since \mathbf{x} is an imputation, we have $x(N) \leq 1$ and $x_i \geq 0$ for all $i \in N$. Now, by non-negativity of \mathbf{x} and by the fact that $x(S) \leq 1$ for all $S \subseteq N$, the term on the left can be rewritten as

$$\begin{aligned} & \min\{x(S) - v(S) \mid S \in 2^N \setminus \{N, \emptyset\}\} \\ &= \min\{x(S) - 1 \mid S \in \mathcal{M}\}, \end{aligned}$$

where \mathcal{M} is the set of all minimal winning coalitions in $2^N \setminus \{N, \emptyset\}$. It remains to show that the computation of $\min\{x(S) - 1 \mid S \in \mathcal{M}\}$ is in FPT with respect to the supermodular degree. Fix $i \in N$ and let $\mathcal{M}(i)$ denote the set of all coalitions in \mathcal{M} including i . By Corollary 1, these coalitions are subsets of $D^+(i) \cup \{i\}$, i.e., $\mathcal{M}(i) \subseteq 2^{D^+(i) \cup \{i\}}$. By iterating through all $i \in N$ and all subsets of $D^+(i) \cup \{i\}$, we can compute the value $\min\{x(S) - 1 \mid S \in \mathcal{M}\}$. \square

When the dependency graph has degree at most 2, Feige and Izsak (2013) showed that demand queries can be answered in polynomial time; that is, given a characteristic function $v : 2^N \rightarrow \mathbb{R}$ and a vector $\mathbf{x} \in \mathbb{R}^N$, one can efficiently compute a subset S^* maximizing $v(S^*) - x(S^*)$ over any subfamily of 2^N . This allows us to obtain the following result.

Theorem 6. *Consider a game (N, v) whose dependency degree is at most 2. Given oracle access to the characteristic function v , we can decide the non-emptiness of the core or find an element of the least core in time polynomial in n .*

Proof. We will argue that the separation problem for LP_0 can be solved in time polynomial in n . Consider an $\varepsilon \in \mathbb{R}$ and a vector $\mathbf{x} \in \mathbb{R}^N$. Again, one can decide whether \mathbf{x} is

an imputation in time $O(n)$. Now, if the dependency degree is at most 2, it follows from the work of Feige and Izsak (2013) that the maximum value $v(S) - x(S)$ over coalitions in $S \in 2^N \setminus \{N, \emptyset\}$ can be computed in time polynomial in n ; thus, we can compare the maximum with the given ε and efficiently decide whether $(\varepsilon, \mathbf{x})$ satisfies the inequalities in LP_0 .

It remains to notice that the imputations in the core are exactly the optimal solutions to LP_0 with ε replaced with the value 0, i.e., we can use the same procedure as above to decide whether the core is non-empty. \square

However, this argument does not extend to general cooperative games: we will now demonstrate that there is a succinctly representable class of games for which the separation problem for LP_0 is NP-hard, even though the dependency degree of games in this class is bounded by a small constant and their supermodular degree is 1,

Theorem 7. *There exists a family of hypergraph games with dependency degree 7 and supermodular degree 1 for which the separation problem for LP_0 is NP-hard.*

Proof. An instance of 3-REGULAR INDEPENDENT SET is given by a 3-regular graph G and an integer k ; it is a ‘yes’-instance if G has an independent set of size at least k and a ‘no’-instance otherwise. This problem is known to be NP-hard (Garey and Johnson 1979). We will now show how to reduce it to the separation problem for LP_0 for a family of hypergraph games defined below. For every vertex $v \in V$, we introduce two players v^1, v^2 , and let $N = \{v^1, v^2 \mid v \in V\} \cup \{d^1, d^2\}$. We define a set function v by building its hypergraph representation $H = (N, E_H, w)$, so that for each $S \subseteq N$ the value $v(S)$ is computed as the sum of the weights of all hyperedges of the sub-hypergraph induced by S . We set $w(u) = 0$ for each $u \in N$. For each $v \in V$ we connect v_1 and v_2 by an edge (i.e., a hyperedge of rank 2) of weight 3, and we connect d^1 and d^2 by an edge of weight $|V|/2$. For every edge $e = \{u, v\} \in E$, we introduce a hyperedge of size 4 and weight -1 containing u^1, u^2, v^1 and v^2 . Since G is 3-regular, $|E| = \frac{3}{2}|V|$ and hence $v(N) = 3|V| - \frac{3}{2}|V| + \frac{1}{2}|V| = 2|V|$. Moreover, the constructed game has dependency degree $d = 7$ since G is 3-regular, and there is a pair of dependent vertices for every vertex in G . Also, the supermodular degree p of this game is 1, as the supermodular dependency set of every vertex only contains the other vertex belonging to the same pair.

Now, we set $x_u^* = 1$ for each $u \in N \setminus \{d^1, d^2\}$ and $x_{d^1}^* = x_{d^2}^* = 0$. Clearly, the vector \mathbf{x}^* is an imputation, as we have $v(N) = \mathbf{x}^*(N) = 2|V|$ and the constraint $x_i^* \geq v(\{i\})$ is satisfied for every $i \in N$.

It can be shown that G admits an independent set of size k if and only if the maximum excess at \mathbf{x}^* is at least $k + \frac{|V|}{2}$. Indeed if $I \subseteq V$ is an independent set of size k in G , then $\{v^1, v^2 : v \in I\} \cup \{d^1, d^2\}$ is a coalition whose excess at \mathbf{x}^* is $k + |V|/2$. Conversely, it can be argued that if there is a coalition whose excess at \mathbf{x}^* is $k + \frac{|V|}{2}$ then G has an independent set of size k . \square

5 Complexity of the Shapley and Banzhaf Values

For the Shapley and Banzhaf values, the following observation is crucial for our analysis: for every player i and every coalition $C \subseteq N \setminus \{i\}$ it holds that adding players who do not depend on i to C does not affect i 's marginal contribution to C . We formalize this observation in the following lemma. We write $I(i) = N \setminus (D(i) \cup \{i\})$.

Lemma 3. *For every $S \subseteq D(j)$ and for every $T \subseteq I(j)$ it holds that $v(j|S \cup T) = v(j|S)$.*

Proof. We prove our claim by induction on the size of the set T . The claim clearly holds when $|T| = 1$. Suppose it holds for $|T| \leq k - 1$. Let $T = \{i_1, i_2, \dots, i_k\}$. Then $v(j|S \cup \{i_1, i_2, \dots, i_{k-1}\}) = v(j|S)$ by the induction hypothesis and $v(j|S \cup \{i_1, i_2, \dots, i_{k-1}, i_k\}) = v(j|S \cup \{i_1, i_2, \dots, i_{k-1}\})$ by the fact that $i_k \in I(j)$. Combining these two equations yields

$$v(j|S \cup T) = v(j|S \cup \{i_1, i_2, \dots, i_{k-1}, i_k\}) = v(j|S). \quad \square$$

By Lemma 3, one can calculate the Shapley value of each player by iterating over all subsets of her dependency set, computing her marginal contribution, and counting the number of coalitions whose intersection with the dependency set is exactly this subset.

Theorem 8. *Computing the Shapley value is in FPT with respect to the dependency degree.*

Proof. The Shapley value can be written as follows:

$$\begin{aligned} \phi_i(N, v) &= \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(n - |S| - 1)!}{n!} v(i|S), \\ &= \frac{1}{n!} \sum_{S \subseteq D(i)} \sum_{T \subseteq I(i)} \frac{|S \cup T|!(n - |S \cup T| - 1)!}{n!} v(i|S \cup T), \\ &= \frac{1}{n!} \sum_{S \subseteq D(i)} v(i|S) \sum_{t=0}^{n-|S|-1} \alpha(S, t), \end{aligned}$$

where $\alpha(S, t) = \binom{n-|S|-1}{t} \frac{(|S|+t)!(n-|S|-t-1)!}{n!}$. The last equality holds due to Lemma 3. \square

Similarly, the Banzhaf value can be efficiently computed when the dependency degree is bounded.

Theorem 9. *Computing the Banzhaf value is in FPT with respect to the dependency degree.*

Proof. The Banzhaf value can be written as follows:

$$\begin{aligned} \beta_i(N, v) &= \frac{1}{2^{n-1}} \sum_{S \subseteq N \setminus \{i\}} v(i|S), \\ &= \frac{1}{2^{n-1}} \sum_{S \subseteq D(i)} \sum_{T \subseteq I(i)} v(i|S \cup T), \\ &= \frac{1}{2^{n-1}} \sum_{S \subseteq D(i)} v(i|S) 2^{(n-|S|-1)}, \end{aligned}$$

where the last equality holds due to Lemma 3. \square

6 Optimal Coalition Structure Generation

So far, we discussed games *without* coalition structures: we implicitly assume that all players cooperate and are willing to divide the value of the grand coalition. In some settings, however, it may be more efficient to split the players into different teams. The problem of finding the best partition of players, which is referred to as the *optimal coalition structure generation problem*, has thus been extensively studied (see, e.g., the surveys by Elkind, Rahwan, and Jennings (2013) and Rahwan et al. (2015)). Formally, a *coalition structure* for N is a partition $\pi = \{S_1, S_2, \dots, S_\ell\}$ of N into disjoint coalitions. A coalition structure π for N is *optimal* if it maximizes the *social welfare* $\sum_{S \in \pi} v(S)$.

Not surprisingly, optimal coalition structure generation is NP-hard even for weighted voting games; this can be shown by a straightforward reduction from PARTITION (Chalkiadakis, Elkind, and Wooldridge 2011). In contrast, we will now argue that if the input game is simple and its supermodular dependency graph has tree-like structure, this problem becomes tractable. To this end, we introduce the notions of *tree decomposition* and *treewidth*.

Definition 1. A *tree decomposition* of a graph G is a pair $(T, (X_t)_{t \in V(T)})$, where T is a rooted tree and $(X_t)_{t \in V(T)}$ is a family of subsets of $V(G)$, called *bags*, where

- (i) for every node $i \in V(G)$, the set $X^{-1}(i) := \{t \in V(T) \mid i \in X_t\}$ is nonempty and connected in T , and
- (ii) for every edge $\{i, j\} \in E(G)$, there is a node $t \in V(T)$ such that $i, j \in X_t$.

For a node t of T , we let V_t be the union of all bags present in the subtree of T rooted at t , including X_t . The *treewidth* of a tree decomposition $(T, (X_t)_{t \in V(T)})$ of G is $\max_{t \in V(T)} (|X_t| - 1)$.

Recall that for simple games it holds that minimal winning coalitions form cliques in the supermodular dependency graph. We will now argue that this implies that there is an optimal coalition structure where each winning coalition is contained in some bag of the tree decomposition.

Lemma 4. *Let (N, v) be a simple game and let $(T, (X_t)_{t \in V(T)})$ be a tree decomposition of the supermodular dependency graph G_v^+ . For every $t \in V(T)$ it holds that every optimal coalition structure π for V_t can be transformed into another optimal coalition structure π' so that for each coalition $S \in \pi'$ the following statements hold:*

- (i) if S contains a player x with $x \in X_t \setminus X_{t'}$ for every child t' of t , then $S \subseteq X_t$.
- (ii) S can appear in only one subtree, i.e., if $S \cap X_t$ is a subset of both X_{t_1} and X_{t_2} for some children t_1, t_2 of t , then $S \subseteq V_{t_1}$ or $S \subseteq V_{t_2}$.

Proof. Consider a node $t \in V(T)$. Note that every optimal coalition structure can be transformed into another optimal coalition structure where each winning coalition is minimal. Thus, let π be such an optimal coalition structure for V_t . Take an arbitrary coalition $S \in \pi$. If S is losing, then it is clear that S can be divided into $S \cap X_t$ and $S \cap (V_{t'} \setminus X_t)$ for each child t' of t without changing the sum $\sum_{S \in \pi} v(S)$,

and hence our claims hold. Now, suppose that S is a minimal winning coalition. By Theorem 3, S forms a clique.

To show (i), suppose that there is a player $x \in S$ such that $x \in X_t \setminus X_{t'}$ for every child t' of t . Assume towards a contradiction that $S \not\subseteq X_t$ and hence there is a player $y \in S \setminus X_t$. Observe that no bag X_w , $w \in V(T)$, contains both x and y , since x is not present in $X_{t'}$ for any successor t' of t , and y does not appear above t . However, since S is a clique, x is adjacent to y in the graph G_v^+ , which means that there is a bag containing both x and y . This contradicts requirement (ii) of Definition 1.

To show (ii), suppose that $S \cap X_t$ is a subset of both X_{t_1} and X_{t_2} for some children t_1, t_2 of t . Assume towards a contradiction that $S \not\subseteq V_{t_1}$ and $S \not\subseteq V_{t_2}$; thus there exist a player $x \in S \setminus V_{t_1}$ and a player $y \in S \setminus V_{t_2}$. Since the intersection $S \cap X_t$ is fully contained in both V_{t_1} and V_{t_2} , we have $x, y \notin X_t$, and hence no bag contains both x and y . However, since S is a clique, x and y are adjacent, contradicting requirement (ii) of Definition 1. \square

By Lemma 4, one can find an optimal coalition structure for a simple game by trying all possible partitions of each bag and combining them in a bottom-up manner. The proof of the following theorem can be found in the full version of the paper (Igarashi, Izsak, and Elkind 2017).

Theorem 10. *Consider a simple game (N, v) . There exists an algorithm that, given oracle access to v and a tree decomposition $(T, (X_t)_{t \in V(T)})$ of the supermodular dependency graph G_v^+ with treewidth τ , computes an optimal coalition structure in time $O((\tau + 1)^{\tau+1} 4^{\tau+1} |V(t)|)$.*

Voice, Polukarov, and Jennings (2012) also studied the problem of finding an optimal coalition structure for a graph-restricted instance, and designed a $O(\tau^{\tau+O(1)} n)$ algorithm for a graph with treewidth τ when players disconnected on a graph do not depend on each other. In our setting, the result of Voice, Polukarov, and Jennings (2012) translates into an efficient algorithm for finding an optimal coalition structure for games whose dependency graph has bounded treewidth. However, this result does not imply ours, since Theorem 10 is for games whose supermodular dependency graph has bounded treewidth, and the supermodular dependency graph is a subgraph of the dependency graph.

Recall that a chordal graph always admits a clique-tree decomposition, i.e., a tree decomposition where each bag forms a maximal clique; moreover, such a decomposition can be found in linear time (Berry and Simonet 2017). Hence the treewidth of a chordal graph is bounded by the maximum degree of the graph, and we have the following corollary.

Corollary 2. *There exists an algorithm that, given a weighted voting game $[N; w; q]$ and its supermodular dependency graph, computes an optimal coalition structure in time $O((p + 1)^{p+1} 4^{p+1} (p + 1)n)$.*

Our FPT result with respect to the supermodular degree does not extend to arbitrary simple games; indeed, for such games we obtain a hardness result that holds even if the supermodular degree is bounded by a constant (see the supplemental material for the proof).

Theorem 11. *Finding an optimal coalition structure of a simple game is NP-hard even if the supermodular degree is at most 6.*

7 Constructing Dependency Graphs

In this section, we explore another question: if the dependency graph is not given to us as input, how hard is it to compute the dependency/supermodular degree?

An interesting class of games where constructing dependency graphs is easy is *induced subgraph games*. Formally, an induced subgraph game with a set of players N is given by an undirected graph $G = (N, E)$ and a weight function $w : E \rightarrow \mathbb{R}$. For each $S \subseteq N$ the value $v(S)$ of a coalition S is given by the sum of edge weights in the graph induced by S . In such games, player i depends on player j if and only if the edge $\{i, j\}$ has non-zero weight; furthermore, their dependence is positive if and only if the weight is positive.

However, for general games constructing the dependency graph is not easy. As we have seen before, a dummy player corresponds to an isolated vertex of both dependency and supermodular dependency graphs; deciding whether a player is a dummy is NP-hard, for instance, in weighted voting games (Chalkiadakis, Elkind, and Wooldridge 2011) or in threshold network flow games (Bachrach and Rosenschein 2008). This immediately implies the hardness of constructing a supermodular dependency graph or a dependency graph for these classes of games. On the positive side, for weighted voting games we can determine if there is a dependency between players in pseudo-polynomial time.

Theorem 12. *Given a weighted voting game $[N; w; q]$ with integer weights and players $i, j \in N$ with $i \neq j$, one can decide whether i depends on j in time polynomial in n and w_{\max} , where w_{\max} is the maximum weight among the players.*

Proof. We need to check if there is a coalition $S \subseteq N \setminus \{i, j\}$ such that (a) coalition $S \cup \{i, j\}$ is winning, but coalitions $S \cup \{i\}$ and $S \cup \{j\}$ are losing, or (b) coalitions $S \cup \{i\}$ and $S \cup \{j\}$ are winning, but S is losing. Note that condition (a) is equivalent to $q - (w_i + w_j) \leq w(S) < \min\{q - w_i, q - w_j\}$, and condition (b) is equivalent to $\max\{q - w_i, q - w_j\} \leq w(S) < q$. Both conditions can be checked by considering at most w_{\max} weights, and, for each weight, checking if $N \setminus \{i, j\}$ contains a coalition of that weight; the latter question is effectively an instance of SUBSET SUM and hence can be answered in pseudo-polynomial time. \square

One can also ask if it is possible to construct the dependency graph by evaluating the characteristic function at polynomially many points. However, perhaps unsurprisingly, the query complexity of the dependency graph turns out to be exponential in the number of players.

Theorem 13. *Any algorithm that computes the dependency graph of a simple game must evaluate the characteristic function in at least $\binom{n}{n/2}$ points in the worst case.*

Proof. We will describe a class of simple games where $\binom{n}{n/2}$ value queries are required to decide whether a player is a dummy.

Let n be an even number, and set $X = [n]$. We introduce one new player $a \notin X$. For each subset $H \subseteq [n]$ with $|H| = n/2$, we construct a simple game $\mathcal{G}_H = (N, v_H)$. The player set in this game is $N = X \cup \{a\}$, and $v_H(S) = 1$ if and only if $|S \cap X| \geq \frac{n}{2}$ and $S \neq H$ or if $S = H \cup \{a\}$. Notice that player a is not a pivot for any coalition in \mathcal{G}_H except for H . We also define a game $\mathcal{G}_0 = (N, v_0)$, where $N = X \cup \{a\}$ and $v_0(S) = 1$ if and only if $|S \cap X| \geq \frac{n}{2}$. Clearly, a is a dummy in \mathcal{G}_0 .

Consider any algorithm that constructs a dependency graph using value queries. If it asks fewer than $\binom{n}{n/2}$ queries, then there is a coalition $H \subseteq X$ with $|H| = n/2$ whose value has not been queried. Now suppose that whenever the algorithm queries the value of a coalition of size $n/2$, the answer is 1. Then at the end the algorithm is unable to distinguish between \mathcal{G}_H and \mathcal{G}_0 , and hence it is unable to decide whether a is a dummy. \square

8 Conclusions

In this work, we have demonstrated that the concepts of dependency degree and supermodular degree are useful in the analysis on cooperative games: we have obtained FPT results with respect to these parameters for a number of solution concepts in cooperative game theory. We have also explored the limitations of this approach, proving hardness results for games where these parameters are bounded by a constant. In the future, it would be interesting to extend this line of work to other solution concepts, such as the nucleolus, the kernel, or the bargaining set, or to other succinctly representable classes of cooperative games.

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