Condorcet Relaxation In Spatial Voting

Arnold Filtser, [1] **Omrit Filtser** [2]
[1] Columbia University
[2] Stony Brook University
arnold@filtser.com, omrit@filtser.com

Abstract

Consider a set of voters V, represented by a multiset in a metric space (X, d). The voters have to reach a decision - a point in X. A choice $p \in X$ is called a β-plurality point for V, if for any other choice $q \in X$ it holds that $|\{v \in V \mid \beta \cdot d(p, v) \leq d(q, v)\}| \geq \frac{|V|}{2}$. In other words, at least half of the voters "prefer" p over q, when an extra factor of β is taken in favor of p. For $\beta = 1$, this is equivalent to Condorcet winner, which rarely exists. The concept of β-plurality was suggested by Aronov, de Berg, Gudmundsson, and Horton [SoCG 2020] as a relaxation of the Condorcet criterion.

Denote by $\beta^*_{(X,d)}$ the value $\sup\{\beta \mid$ every finite multiset V in X admits a β-plurality point$\}$. The parameter β^* determines the amount of relaxation required in order to reach a stable decision. Aronov et al. showed that for the Euclidean plane $\beta^*_{(\mathbb{R}^2,\|\cdot\|_2)} = \frac{\sqrt{3}}{2}$, and more generally, for d-dimensional Euclidean space, $\frac{1}{\sqrt{d}} \leq \beta^*_{(\mathbb{R}^d,\|\cdot\|_2)} \leq \frac{\sqrt{3}}{2}$. In this paper, we show that $0.557 \leq \beta^*_{(\mathbb{R}^d,\|\cdot\|_2)}$ for any dimension d (notice that $\frac{1}{\sqrt{d}} < 0.557$ for any $d \geq 4$). In addition, we prove that for every metric space (X, d) it holds that $\sqrt{2} - 1 \leq \beta^*_{(X,d)}$, and show that there exists a metric space for which $\beta^*_{(X,d)} \leq \frac{1}{2}$.

Introduction

When a group of agents wants to reach a joint decision, it is often natural to embed their preferences in some metric space. The preferences of each agent are represented by a metric point (also referred to as a voter). Each point in the metric space is a potential choice, where an agent/voter prefers choices closer to its point over farther choices. The goal is to reach a stable decision, in the sense that no alternative choice is preferred by a majority of the voters. Such a decision is often referred to as a Condorcet winner.

More formally, consider a metric space (X, d), and a finite multiset of points V from X, called voters. A voter v prefers a choice $p \in X$ over a choice $q \in X$ if $d(p, v) < d(q, v)$. Specifically, a choice point $p \in X$ is a *plurality point* if for any other choice point $q \in X$, the number of voters preferring p over q is at least the number of voters preferring q over p, i.e., $|\{v \in V \mid d(p, v) < d(q, v)\}| \geq |\{v \in V \mid d(p, v) > d(q, v)\}|$. [1]

The special case where (X, d) is the Euclidean space, i.e., $(\mathbb{R}^d, \|\cdot\|_2)$, is called spatial voting games, and was studied in the political economy context (Black 1948; Downs 1957; Plott 1967; Enelow and Hinich 1983). When $X = \mathbb{R}$ is the real line, a plurality point always exists, in fact, it is simply the median of V. When (X, d) is induced by the shortest path metric of a tree graph, then again a plurality point always exists, as any separator vertex [2] is a plurality point. However, already in $\mathbb{R}^2$ a plurality point does not always exist, and moreover, it exists only for a negligible portion of the point sets. Indeed, for any $d \geq 2$, a plurality point for a multiset V in $\mathbb{R}^d$ exists if and only if all median hyperplanes [3] for V have a common intersection point (see (Enelow and Hinich 1983; Plott 1967)). Wu et al. (2014) and de Berg et al. (2018) presented algorithms that determine whether such a point exist.

Recently, Aronov, de Berg, Gudmundsson, and Horton (2020), introduced a relaxation for the concept of plurality points, by defining a point $p \in X$ to be a *β-plurality point*, for $\beta \in (0, 1]$, if for every other point $q \in X$, $|\{v \in V \mid \beta \cdot d(p, v) < d(q, v)\}| \geq |\{v \in V \mid \beta \cdot d(p, v) > d(q, v)\}|$. In other words, if we scale distances towards p by a factor of β, then for every choice point q, the number of voters preferring p over q is at least the number of voters preferring q over p. Set

$$\begin{aligned}
\beta_{(X,d)}(p, V) &= \sup\{\beta \mid p \text{ is a } \beta\text{-plurality point in } X \text{ w.r.t. } V\}, \\
\beta_{(X,d)}(V) &= \sup_{p \in X}\{\beta_{(X,d)}(p, V)\}, \\
\beta^*_{(X,d)} &= \inf\{\beta_{(X,d)}(V) \mid V \text{ is a multiset in } X\} .
\end{aligned}$$

[1] A more accurate name for such a point, which is also used in the literature, is *Condorcet winner*. However, as this work is mainly concerned with the term β-plurality point defined in (Aronov et al. 2020), we choose to keep their terminology.

[2] If T is the tree inducing (X, d), a separator vertex is a vertex $z \in X$, the removal of which will break the graph $T \setminus \{z\}$ into connected components, each containing at most $\frac{|V|}{2}$ voters. Every tree contains a separator vertex (Jordan 1869).

[3] A median hyperplane for V is a hyperplane such that both open half-spaces defined by it contain less than $\frac{|V|}{2}$ voters.

Space	Lower Bound	Upper Bound	Ref
$\mathbb{R}$ and tree metric	1	1	
$(\mathbb{R}^2, \|\cdot\|_2)$	$\sqrt{3}/2 \approx 0.866$	$\sqrt{3}/2$	(Aronov et al. 2020)
$(\mathbb{R}^3, \|\cdot\|_2)$	$1/\sqrt{3} \approx 0.577$	$\sqrt{3}/2$	(Aronov et al. 2020)
$(\mathbb{R}^d, \|\cdot\|_2)$ **for** $d \geq 4$	≈ 0.557	$\sqrt{3}/2$	Theorem 4, (Aronov et al. 2020)
General metric space	$\sqrt{2} - 1 \approx 0.414$	$1/2$	Theorem 1, Theorem 2

Table 1: Summary of the state of the art results on β^*_X for different metric spaces.

Intuitively, $\beta_{(X,d)}(p, V)$ is the maximum value β such that p is a β-plurality point, $\beta_{(X,d)}(V)$ is the maximum value β such that there exist a β-plurality point in (X, d) w.r.t. the voter set V, and $\beta^*_{(X,d)}$ is the maximum value β such that for every voter set V, a β-plurality point is guaranteed to exist.

A natural question is to find or estimate these parameters for a given metric space. Notice that as β becomes larger, we are "closer" to having a plurality point. Therefore, it is interesting to know what values of β we can anticipate for a given metric space in order to reach a stable decision. These bounds give an indication on the amount of relaxation that might be needed, and how reasonable it is.

Aronov et al. (2020) studied β-plurality for the case of Euclidean space, i.e., $(\mathbb{R}^d, \|\cdot\|_2)$. Given a specific instance V, they presented an EPTAS to approximate $\beta_{(\mathbb{R}^d,\|\cdot\|_2)}(V)$. For the case of the Euclidean plane ($d = 2$), they showed that $\beta^*_{(\mathbb{R}^2,\|\cdot\|_2)} = \frac{\sqrt{3}}{2}$. Specifically, they showed that for every multiset of voters V in $\mathbb{R}^2$, there exists a point $p \in \mathbb{R}^2$ such that $\beta_{(\mathbb{R}^2,\|\cdot\|_2)}(V, p) \geq \frac{\sqrt{3}}{2}$. Furthermore, they showed that for the case where V consist of the three vertices of an equilateral triangle, it holds that $\beta_{(\mathbb{R}^2,\|\cdot\|_2)}(V) \leq \frac{\sqrt{3}}{2}$. For the general d-dimensional Euclidean space $(\mathbb{R}^d, \|\cdot\|_2)$, Aronov et al. showed a lower bound of $\beta^*_{(\mathbb{R}^2,\|\cdot\|_2)}(V) \geq \frac{1}{\sqrt{d}}$. They left as a "main open problem" the question of closing the gap between $\frac{1}{\sqrt{d}}$ and $\frac{\sqrt{3}}{2}$, and asked what bound on β^* could be proven in other metric spaces.

Our contribution. We prove that for every metric space (X, d), $\beta^*_{(X,d)} \geq \sqrt{2}-1$. Note that Aronov et al. (2020) gave a lower bound of $\frac{1}{\sqrt{d}}$ for the Euclidean metric, while our result shows a constant lower bound for any metric space. In addition, we provide an example of a metric space (X, d) for which $\beta^*_{(X,d)} = \frac{1}{2}$. In fact, we show that $\beta^*_{(X,d)} = \frac{1}{2}$ for any (continuous) graph metric (X, d) that contains a cycle (in contrast to tree metrics, for which $\beta^*_{(X,d)} = 1$). Finally, for the case of Euclidean space of arbitrary dimension d, we show that $\beta^*_{(\mathbb{R}^d,\|\cdot\|_2)} \geq 0.557$. Note that this lower bound is larger than $\frac{1}{\sqrt{d}}$ for $d \geq 4$. All the current and previous results are summarized in Table 1.

Related work A well known relaxation for the concept of plurality points in Euclidean space is the yolk (McKelvey 1986; Miller, Grofman, and Feld 1989; Feld, Grofman, and Miller 1988; Gudmundsson and Wong 2019; Miller 2015), which is the smallest ball intersecting every median hyperplane [3] of V. The center of the yolk is a good heuristic for a plurality point (see (Miller and Godfrey 2008) for a list of properties the yolk possesses). Notice that the definition of β-plurality applies for any metric space, not necessarily Euclidean as in the concept of yolk.

Another relaxation studied by Lin et al. (2015) is the "minimum cost plurality problem". Here given a set of voters V with some cost function, the goal is to find a set W of minimum cost such that $V \setminus W$ contains a plurality point.

A main drawback of the spatial voting model in the realistic political context was underlined by Stokes (1963). The claim is that this model does not take into account the so-called "valence issues": qualities of the candidates such as charisma and competence (Evrenk et al. 2018), a strong party support (Wiseman 2006), and even the campaign spending (Herrera, Levine, and Martinelli 2008). Therefore, several more realistic models have been proposed (see, e.g., (Giansiracusa and Ricciardi 2019; Gouret, Hollard, and Rossignol 2011; Sanders et al. 2011)). A common model is the multiplicative model which was introduced by Hollard and Rossignol (2008), and is defined for two-candidate spatial voting model. This model is closely related to the concept of β-plurality, and is similar to it in the case of a two-player game.

General Metric Spaces

We begin by providing an alternative definition of β-plurality point.

Definition 1. *Consider a metric space (X, d), and a multiset V in X of voters. A point $p \in X$ is a β-plurality point if for every $q \in X$, $|\{v \in V \mid \beta \cdot d(p, v) \leq d(q, v)\}| \geq \frac{|V|}{2}$. In addition, similarly to (Aronov et al. 2020), set $\beta_{(X,d)}(V) = \sup_{p\in X} \beta_{(X,d)}(p, V)$ and*

$$\beta^*_{(X,d)} = \inf\{\beta_{(X,d)}(V) \mid V \text{ is a multiset in } X\} .$$

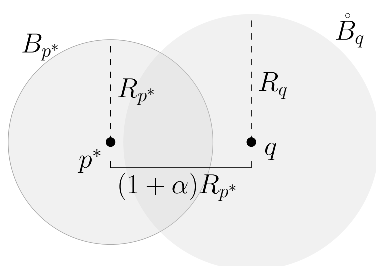
The difference between the definitions is that Definition 1 is deciding ties in favor of p, that is a voter v for which $\beta \cdot d(p, v) = d(q, v)$, will choose p over q, while in the original definition, such voters remain "undecided". Definition 1 is equivalent to the original definition in (Aronov et al. 2020). See Lemma 8 in the "Missing proofs" section for an exact statement and proof.

Consider a metric space (X, d), with a multiset V of voters from X, and set $|V| = n$. For a point p and radius r, denote by $B_V(p, r) = \{v \in V \mid d(p, v) \leq r\}$ the (multi) subset of voters at distance at most r from p (i.e., those that are contained in the closed ball of radius r centered at p), and let R_p be the minimum radius such that $|B_V(p, R_p)| \geq \frac{n}{2}$.

The following theorem shows that a $(\sqrt{2} - 1)$-plurality point always exists. The fact that the lower bound is constant, and even close to $\frac{1}{2}$, demonstrates the strength of β-plurality in the sense that for any set of voters and in any metric space, the multiplication factor needed for the existence of such winner is a fixed constant, and thus the amount of relaxation is bounded.

Theorem 1. *For every metric space (X, d), it hold that $\beta^*_{(X,d)} \geq \sqrt{2} - 1$.*

Proof. Let $p^* \in X$ be a point with minimum R_p over all $p \in X$, and denote $B_{p^*} = B_V(p^*, R_{p^*})$. We claim that p^* is a $(\sqrt{2} - 1)$-plurality point.



Set $\beta = \sqrt{2} - 1$, and notice that $\beta = \frac{1}{2+\beta}$. Consider some choice point $q \in X$, and set $d(p^*, q) = (1 + \alpha) \cdot R_{p^*}$, for $\alpha \geq -1$. Let $\mathring{B}_q = \{v \in V \mid d(q, v) < R_q\}$ be the (multi) subset of voters at distance (strictly) smaller than R_q from q (i.e., those that are contained in the open ball of radius R_q centered at q). Consider the following cases:

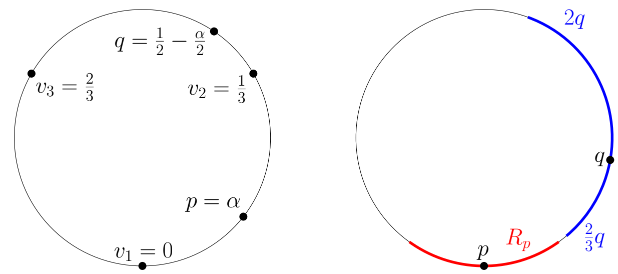
- $\boldsymbol{\alpha \leq \beta}$: For every point $v \notin \mathring{B}_q$, as $d(q, v) \geq R_q \geq R_{p^*}$, by the triangle inequality it holds that
$$\begin{aligned} d(p^*, v) &\leq d(p^*, q) + d(q, v) \leq (2 + \alpha) \cdot d(q, v) \\ &\leq (2 + \beta) \cdot d(q, v) = \frac{1}{\beta} \cdot d(q, v) \ . \end{aligned}$$
- $\boldsymbol{\alpha \geq \beta}$: For every point $v \in B_{p^*}$, as $d(p^*, q) = (1 + \alpha) \cdot R_{p^*} \geq (1 + \alpha) \cdot d(p^*, v)$, it holds that
$$\begin{aligned} d(q, v) &\geq d(q, p^*) - d(p^*, v) \\ &\geq (1 + \alpha - 1) \cdot d(p^*, v) \geq \beta \cdot d(p^*, v) \ . \end{aligned}$$

The theorem follows as $|\mathring{B}_q| < \frac{n}{2} \leq |B_{p^*}|$. □

Theorem 2. *There exists a metric space (X, d) such that $\beta^*_{(X,d)} = \frac{1}{2}$.*

Proof. Consider a continuous cycle C of length 1, we will think of C as a one dimensional space. Formally, X is the segment $[0, 1)$, and given two points $x, y \in [0, 1)$, their distance is $d(x, y) = \min\{(x - y) \bmod 1, (y - x) \bmod 1\}$.

First we show that $\beta^*_{(X,d)} \leq \frac{1}{2}$. Consider a set of three voters $\{v_1, v_2, v_3\} = \{0, \frac{1}{3}, \frac{2}{3}\}$, all at distance $\frac{1}{3}$ from each other. We will show that $\beta^*_{(X,d)}(V) \leq \frac{1}{2}$. Assume by contradiction that there is a choice p which is a β-plurality point for $\beta > \frac{1}{2}$. Assume w.l.o.g. that $p = \alpha \in [0, \frac{1}{6}]$ (see the figure above for illustration), the other cases are symmetric.



Consider the choice point $q = \frac{1}{2} - \frac{\alpha}{2}$ lying on the arc $[v_2, v_3]$ at distance $\frac{1}{6} - \frac{\alpha}{2}$ from v_2, and $\frac{1}{6} + \frac{\alpha}{2}$ from v_3. Then $\beta \cdot d(p, v_2) = \beta \cdot (\frac{1}{3} - \alpha) > \frac{1}{6} - \frac{\alpha}{2} = d(q, v_2)$ and $\beta \cdot d(p, v_3) = \beta \cdot (\frac{1}{3} + \alpha) > \frac{1}{6} + \frac{\alpha}{2} = d(q, v_3)$, which contradicts the assumption that p is a β-plurality point.

Next we show that $\beta^*_{(X,d)} \geq \frac{1}{2}$. Consider an arbitrary (multi) subset of voters $V \subseteq X$, and let $p \in X$ be a choice with minimal radius R_p such that $|B_V(p, R_p)| \geq \frac{n}{2}$. Note that the length of the smallest arc containing $\frac{n}{2}$ voters is $2R_p$. In particular, as either the arc $[0, \frac{1}{2})$ or $[\frac{1}{2}, 1)$ contain $\frac{n}{2}$ voters, $2R_p \leq \frac{1}{2}$, and thus $R_p \leq \frac{1}{4}$. Assume w.l.o.g. that $p = 0$. We show that p is a $\frac{1}{2}$-plurality point. Let $q \in X$ be any other point. We assume that $q \in (0, \frac{1}{2}]$, the case $q \in [\frac{1}{2}, 1)$ is symmetric. We say that a voter v prefers q over p if $\frac{1}{2}d(p, v) > d(q, v)$. It will be enough to show that at most $\frac{n}{2}$ voters prefer q over p. Let v be a voter that prefer p over q. If $v < q$ then $\frac{1}{2}d(p, v) = \frac{1}{2}v$ and $d(q, v) = q - v$, and thus $v > \frac{2}{3}q$. Else, if $v > q$, then $\frac{1}{2}d(p, v) \leq \frac{1}{2}v$ and $d(q, v) = v - q$ (as otherwise the shortest path from v to q goes through p, implying $d(p, v) < d(q, v)$), and therefore $v < 2q$. We conclude that only voters in the arc $(\frac{2}{3}q, 2q)$ prefer q over p. The rest is case analysis:

- If $q < \frac{3}{2}R_p$, then the arc containing the set of the voters preferring q over p is of length $\frac{4}{3}q < 2R_p$. By the minimality of R_p, it contains less than $\frac{n}{2}$ voters.
- If $q \geq \frac{3}{2}R_p$, then the arc $[0, R_p]$ is disjoint from the arc $(\frac{2}{3}q, 2q)$. Moreover, as $q < \frac{1}{2}$, all the voters in the arc $[1 - R_p, 1) \subseteq [\frac{1}{4}, 1)$ will prefer p over q. In particular there are at least $\frac{n}{2}$ voters preferring p over q.

□

Given a weighted graph $G = (V, E, w)$, denote by $\tilde{G}$ its continuous counterpart. If G contains a cycle, we can generalize Theorem 2 to $\tilde{G}$. A proof sketch (and definition of continuous counterpart) is deferred to the “Missing proofs” section. This shows a separation between metric spaces obtained by acyclic graphs (trees) which always contain a plurality point (that is, $\beta^*_{(X,d)} = 1$), to metric spaces obtained by all other graphs, for which $\beta^*_{(X,d)} \le \frac{1}{2}$.

Theorem 3. *For every weighted graph $G = (V, E, w)$ containing a cycle, it holds that $\beta^*_{(\tilde{G},d_{\tilde{G}})} \le \frac{1}{2}$.*

Euclidean Space

In this section we consider the case of the Euclidean metric space, and give a bound on $\beta^*_{(\mathbb{R}^d,\|\cdot\|_2)}$ which is independent of d and greater than $\frac{1}{\sqrt{d}}$ for any $d \ge 4$, thus improving the lower bound of (Aronov et al. 2020) for $d \ge 4$.

Theorem 4. *For Euclidean space of arbitrary dimension, $\beta^*_{(\mathbb{R}^d,\|\cdot\|_2)} \ge \beta$,*
for $\beta = \frac{1}{2}\sqrt{\frac{1}{2} + \sqrt{3} - \frac{1}{2}\sqrt{4\sqrt{3} - 3}} \approx 0.5570157181$

We begin with a structural observation regarding the Euclidean space. The proof is deferred to the “Missing proofs” section.

Claim 2. *Fix a pair of candidates $\vec{a}, \vec{b} \in \mathbb{R}^d$. For any $\beta \in (0, 1)$, the set of all voters $\vec{v} \in V$ that do not β-prefer $\vec{a}$ over $\vec{b}$, i.e., $\left\{\vec{v} \in V \mid \beta \cdot \|\vec{a} - \vec{v}\|_2 > \|\vec{b} - \vec{v}\|_2\right\}$, is equal to the intersection of V with the open ball centered at $\vec{o} = \vec{a} + \frac{1}{1-\beta^2} \cdot (\vec{b} - \vec{a})$ with radius $\beta \cdot \|\vec{o} - \vec{a}\|_2$.*

By the above claim we can conclude:

Corollary 3. *For any $\beta \in (0, 1)$, $\vec{a}$ is a β-plurality point if and only if, for every other point $\vec{o} \in \mathbb{R}^d$, the open ball of radius $\beta \cdot \|\vec{o} - \vec{a}\|_2$ around $\vec{o}$ contains at most $\frac{n}{2}$ voters.*

For the remainder of the section, β is the number defined in Theorem 4 and not a general parameter.

Proof of Theorem 4. Consider a multiset $V \subseteq R^d$ of voters. Let $\vec{p}$ be the point that minimizes $R_{\vec{p}}$. By scaling, we can assume w.l.o.g. that $R_{\vec{p}} = 1$. If $\vec{p}$ is a β-plurality point, then we are done. Otherwise, by Corollary 3 there is a point $\vec{q}$ such that the open ball $B_{\mathbb{R}^d}(\vec{q}, \beta \cdot \|\vec{p} - \vec{q}\|_2)$ contains strictly more than $\frac{n}{2}$ voters. Denote $q = \|\vec{p} - \vec{q}\|_2$. Set $\vec{w} = \vec{p} + \left(\frac{1}{2}(1 - \beta^2)q - \beta + \frac{3}{2q}\right)\frac{\vec{q}-\vec{p}}{\|\vec{q}-\vec{p}\|_2}$, we claim that $\vec{w}$ is a β-plurality point.

First, notice that $q > \frac{1}{\beta}$, as otherwise the open ball of radius $\beta q \le 1$ around $\vec{q}$ contains more than $\frac{n}{2}$ voters, a contradiction to the fact that $R_{\vec{p}} = 1$ is the minimal radius of a closed ball containing at least $\frac{n}{2}$ voters. Second, it must hold that $q < \frac{1}{1-\beta}$, because otherwise $\beta q + 1 \le q$, implying that the ball $B_{\mathbb{R}^d}(\vec{p}, R_{\vec{p}})$ and the open ball $B_{\mathbb{R}^d}(\vec{q}, \beta q)$ are disjoint, a contradiction to the fact that the open ball $B_{\mathbb{R}^d}(\vec{q}, \beta q)$ contains more than $\frac{n}{2}$ voters. Therefore, we conclude that

$$\frac{1}{\beta} < q < \frac{1}{1-\beta} \tag{1}$$

Notice that $\vec{p}$ is a $\frac{1}{2}$-plurality point, as no q satisfies eq. (1).

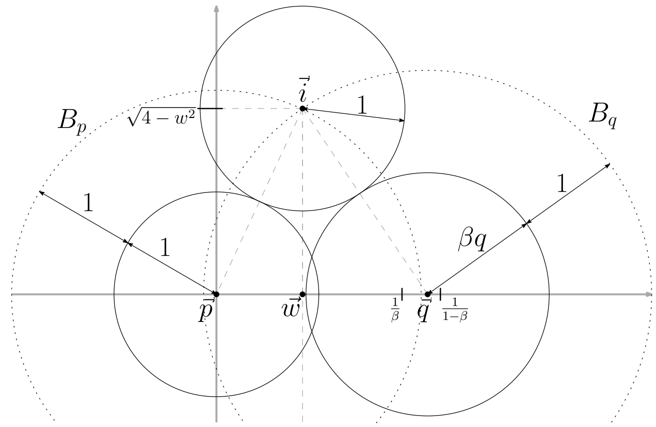


Figure 1: The points $\vec{p} = (0, 0)$, $\vec{q} = (q, 0)$, and $\vec{w} = (w, 0)$ for $w = \frac{1}{2}(1-\beta^2)q-\beta+\frac{3}{2q}$ are on the x-axis. B_p is the circle of radius 2 around $\vec{p}$, while B_q is the circle of radius $1 + \beta q$ around $\vec{q}$. B_p and B_q intersect at $\vec{i} = (w, \sqrt{4 - w^2})$ and $\vec{i'} = (w, -\sqrt{4 - w^2})$. The ball of radius 1 around $\vec{i}$ is tangent to both B_p and B_q. It holds that $\|\vec{w} - \vec{i}\|_2 = \sqrt{4 - w^2} \le \frac{1}{\beta}$ (equation (2)).

To prove that $\vec{w}$ is a β-plurality point, we will show that for every other point $\vec{z} \in \mathbb{R}^d$, the open ball of radius $\beta \cdot \|\vec{z} - \vec{w}\|_2$ around $\vec{z}$ contains at most $\frac{n}{2}$ voters. We will use the following lemma.

Lemma 4. *For any point $\vec{z} \in \mathbb{R}^d$, denote $z = \|\vec{z} - \vec{w}\|_2$. Then at least one of the following hold:*

1. $z \le \frac{1}{\beta}$.
2. $\|\vec{z} - \vec{p}\|_2 \ge 1 + \beta z$.
3. $\|\vec{z} - \vec{q}\|_2 \ge \beta q + \beta z$.

Before proving Lemma 4, we show how it implies that $\vec{w}$ is a β-plurality point. For any $\vec{z} \in \mathbb{R}^d$:

- If $z \le \frac{1}{\beta}$, then $\beta z \le 1 = R_{\vec{p}}$, and thus $B_{\mathbb{R}^d}(\vec{z}, \beta z)$ contains at most $\frac{n}{2}$ voters.
- If $\|\vec{z} - \vec{p}\|_2 \ge 1 + \beta z$, then the balls $B_{\mathbb{R}^d}(\vec{p}, 1)$ and $B_{\mathbb{R}^d}(\vec{z}, \beta z)$ are disjoint, and thus $B_{\mathbb{R}^d}(\vec{z}, \beta z)$ contains at most $\frac{n}{2}$ voters.
- If $\|\vec{z} - \vec{q}\|_2 \ge \beta q + \beta z$, then the balls $B_{\mathbb{R}^d}(\vec{q}, \beta q)$ and $B_{\mathbb{R}^d}(\vec{z}, \beta z)$ are disjoint, and thus $B_{\mathbb{R}^d}(\vec{z}, \beta z)$ contains at most $\frac{n}{2}$ voters.

We conclude that for every $\vec{z} \in \mathbb{R}^d$, $B_{\mathbb{R}^d}(\vec{z}, z)$ contains at most $\frac{n}{2}$ voters, and thus by Corollary 3, $\vec{w}$ is a β-plurality point. $\square$

Proof of Lemma 4. The points $\vec{p}, \vec{q}, \vec{w}$ lie on a single line. Given an additional point $\vec{z}$, the four points lie on a single plane. Thus, w.l.o.g. we can restrict the analysis to the Euclidean plane. Moreover, we can assume that $\vec{p} = (0, 0)$, $\vec{q} = (q, 0)$, $\vec{w} = (w, 0)$ for $w = \frac{1}{2}(1 - \beta^2)q - \beta + \frac{3}{2q}$, and that $\vec{z} = (z_x, z_y)$ where $z_y \geq 0$ (the case of $z_y \leq 0$ is symmetric).

Denote $B_p = B_{\mathbb{R}^2}(\vec{p}, 2)$ and $B_q = B_{\mathbb{R}^2}(\vec{q}, 1+\beta q)$ (see Figure 1). The boundaries of B_p and B_q intersect at the points $(w, \pm\sqrt{4-w^2})$ (this is the reason for our choice of w). Denote $\vec{i} = (w, \sqrt{4-w^2})$, and notice that $0 < w < q$ for any $q \geq \frac{1}{\beta}$ (this can be verified by straightforward calculations). Lemma 4 follows by the two following claims:

Claim 5. *If $\vec{z} \in B_p \cap B_q$ then $\|\vec{z} - \vec{w}\|_2 \leq \frac{1}{\beta}$.*

Claim 6. *If $\vec{z} \notin B_p \cap B_q$ then either $\|\vec{z} - \vec{p}\|_2 \geq 1 + \beta z$ or $\|\vec{z} - \vec{q}\|_2 \geq \beta q + \beta z$.*

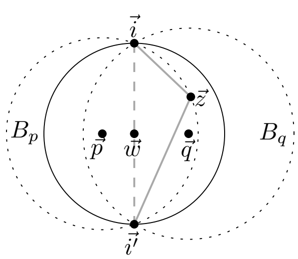
□

Proof of Claim 5. The boundaries of B_p and B_q intersect at the points $\vec{i} = (w, \sqrt{4-w^2})$ and $\vec{i'} = (w, -\sqrt{4-w^2})$. For every $q \in (\frac{1}{\beta}, \frac{1}{1-\beta})$, it holds that

$$\|\vec{i} - \vec{w}\|_2 = \sqrt{4-w^2} \leq \frac{1}{\beta}\,. \tag{2}$$

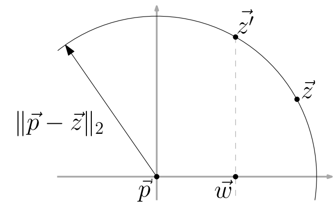
In fact, β was chosen to be the maximum number satisfying equation (2). A calculation showing that equation (2) holds is deferred to the "Missing proofs" section. Consider the ball $B_w = B_{\mathbb{R}^2}(\vec{w}, \|\vec{i} - \vec{w}\|_2)$. B_w has radius at most $\frac{1}{\beta}$, and the segment $[\vec{i}, \vec{i'}]$ is a diameter of B_w. Furthermore, $[\vec{i}, \vec{i'}]$ is a chord in both B_p and B_q.

Assume that $\vec{z} = (z_x, z_y) \in B_p \cap B_q$. If $z_x \geq w$, then the chord $[\vec{i}, \vec{i'}]$ of B_p separates the point $\vec{z}$ from the center $\vec{p}$, because $0 < w < q$ (see illustration below).



It follows that the angle $\angle\vec{i}\vec{z}\vec{i'}$ is larger than $\frac{\pi}{2}$, which implies that $\vec{z} \in B_w$ (as $[\vec{i}, \vec{i'}]$ is a diameter, for any point $\vec{z} \notin B_w$, the angle $\angle\vec{i}\vec{z}\vec{i'}$ is smaller than $\frac{\pi}{2}$). If the $z_x < w$, a symmetric argument (using B_q) will imply that $\vec{z} \in B_w$. We conclude that in any case $\vec{z} \in B_w$. By equation (2), it follows that $\|\vec{z} - \vec{w}\|_2 \leq \frac{1}{\beta}$. □

Proof of Claim 6. Assume that $\vec{z} = (z_x, z_y) \notin B_p \cap B_q$. We show that if $z_x \geq w$ then $\|\vec{z} - \vec{p}\|_2 \geq 1 + \beta z$, and otherwise $\|\vec{z} - \vec{q}\|_2 \geq \beta q + \beta z$.
First, consider the case when $z_x \geq w$. Notice that $\vec{z} \notin B_p$, because the boundaries of B_p and B_q intersect only at $\vec{i}, \vec{i'}$, and thus the intersection of B_p with the half plane $x \geq w$ is contained in B_q. Let $\vec{z'} = (z'_x, z'_y)$ be a point on the ball with radius $\|\vec{z} - \vec{p}\|_2$ around $\vec{p}$ such that $z'_x = w$ and $z'_y \geq 0$, and notice that $z'_y \geq z_y$ (see illustration below).



Notice that $\|\vec{z'} - \vec{w}\|_2 \geq \|\vec{z} - \vec{w}\|_2$, because $z_x^2 + z_y^2 = \|\vec{z} - \vec{p}\|_2^2 = \|\vec{z'} - \vec{p}\|_2^2 = w^2 + {z'_y}^2$ and $z_x \geq w$, so we get $\|\vec{z} - \vec{w}\|_2^2 = z_y^2 + (z_x - w)^2 = z_y^2 + z_x^2 - 2wz_x + w^2 = 2w^2 - 2wz_x + {z'_y}^2 \leq {z'_y}^2 = \|\vec{z'} - \vec{w}\|_2^2$.

Since $\|\vec{z} - \vec{p}\|_2 = \|\vec{z'} - \vec{p}\|_2$, it is enough to show that $\|\vec{z'} - \vec{p}\|_2 \geq 1 + \beta\|\vec{z'} - \vec{w}\|_2$. From here on, we will abuse notation and refer to z' as z. Thus we simply assume $\vec{z} = (w, z)$.

As B_p and B_q intersect at $\vec{i}$, and $\vec{z} \notin B_p \cap B_q$, it must hold that $z \geq \sqrt{4-w^2}$. Note that $\|\vec{p} - \vec{i}\|_2 = 2$ (because $\vec{i}$ is on the boundary of B_p), and by equation (2), $\beta \cdot \|\vec{i} - \vec{w}\|_2 \leq 1$. It thus follows that $1 + \beta\|\vec{w} - \vec{i}\|_2 \leq 2 = \|\vec{p} - \vec{i}\|_2$, implying that the claim holds for $\vec{z} = \vec{i}$. It remains to prove that the claim holds for $\vec{z} = (w, \sqrt{4-w^2} + \delta)$ for all $\delta \geq 0$. It holds that

$$\begin{aligned}\|\vec{z} - \vec{p}\|_2^2 &= w^2 + (\sqrt{4-w^2} + \delta)^2 \\ &= \|\vec{i} - \vec{p}\|_2^2 + 2\delta\sqrt{4-w^2} + \delta^2\,.\end{aligned}$$

$$\begin{aligned}&(1 + \beta\|\vec{z} - \vec{w}\|_2)^2 \\ &= \left(1 + \beta\|\vec{i} - \vec{w}\|_2 + \beta\|\vec{z} - \vec{i}\|_2\right)^2 \\ &= \left(1 + \beta\|\vec{i} - \vec{w}\|_2\right)^2 + 2\beta\|\vec{z} - \vec{i}\|_2\left(1 + \beta\|\vec{i} - \vec{w}\|_2\right) \\ &\qquad + \beta^2\|\vec{z} - \vec{i}\|_2^2 \\ &= \left(1 + \beta\|\vec{i} - \vec{w}\|_2\right)^2 + 2\beta\delta\left(1 + \beta\sqrt{4-w^2}\right) + \beta^2\delta^2\,.\end{aligned}$$

As $1 + \beta\|\vec{w} - \vec{i}\|_2 \leq \|\vec{p} - \vec{i}\|_2$, it holds that

$$\begin{aligned}&\|\vec{z} - \vec{p}\|_2^2 - (1 + \beta\|\vec{z} - \vec{w}\|_2)^2 \\ &\geq \left(2\delta\sqrt{4-w^2} + \delta^2\right) \\ &\qquad - \left(2\beta\delta\left(1 + \beta\sqrt{4-w^2}\right) + \beta^2\delta^2\right) \\ &= 2\delta\sqrt{4-w^2}\left(1 - \beta^2\right) + \delta^2(1 - \beta^2) - 2\beta\delta \geq 0\,,\end{aligned}$$

where the last inequality holds as by our choice of β, $\sqrt{4-w^2}\left(1 - \beta^2\right) \geq \beta$ for every $\frac{1}{\beta} < q < \frac{1}{1-\beta}$. The claim follows.

Next, we show that in the symmetric case, when $z_x \leq w$, it holds that $\|\vec{z} - \vec{q}\|_2 \geq \beta q + \beta z$. Similarly to the previous case, we can assume that $\vec{z} = (w, z)$, where $z \geq \sqrt{4-w^2}$

(as this is only harder). Now, as $\vec{i}$ lies on the boundary of B_q, by equation (2), it holds that $\|\vec{i}-\vec{q}\|_2 = 1+\beta q \geq \beta\|\vec{w}-\vec{i}\|_2 + \beta q$. It remains to prove that the claim holds for $\vec{z} = (w, \sqrt{4-w^2}+\delta)$ for some $\delta > 0$. It holds that

$$\begin{aligned}\|\vec{z}-\vec{q}\|_2^2 &= (q-w)^2 + (\sqrt{4-w^2}+\delta)^2 \\ &= \|\vec{i}-\vec{q}\|_2^2 + 2\delta\sqrt{4-w^2}+\delta^2 .\end{aligned}$$

$$\begin{aligned}&(\beta q + \beta\cdot\|\vec{z}-\vec{w}\|_2)^2 \\ &= \left(\beta q + \beta\cdot\|\vec{i}-\vec{w}\|_2 + \beta\cdot\|\vec{z}-\vec{i}\|_2\right)^2 \\ &= \left(\beta q + \beta\|\vec{i}-\vec{w}\|_2\right)^2 \\ &\qquad + 2\beta\|\vec{z}-\vec{i}\|_2\left(\beta q + \beta\|\vec{i}-\vec{w}\|_2\right) + \beta^2\|\vec{z}-\vec{i}\|_2^2 \\ &\geq \left(\beta q + \beta\|\vec{i}-\vec{w}\|_2\right)^2 \\ &\qquad + 2\beta\delta\left(\beta q + \beta\sqrt{4-w^2}\right) + \beta^2\delta^2 .\end{aligned}$$

Thus,

$$\begin{aligned}&\|\vec{z}-\vec{q}\|_2^2 - (\beta q + \beta\cdot\|\vec{z}-\vec{w}\|_2)^2 \\ &\geq \left(2\delta\sqrt{4-w^2}+\delta^2\right) - \Big(2\beta\delta\left(\beta q + \beta\sqrt{4-w^2}\right) \\ &\qquad\qquad + \beta^2\delta^2\Big) \\ &= 2\delta\sqrt{4-w^2}\left(1-\beta^2\right) + \delta^2(1-\beta^2) - 2\beta^2 q\delta \geq 0 ,\end{aligned}$$

where the last inequality holds as by our choice of β, $\sqrt{4-w^2}\left(1-\beta^2\right) \geq \beta^2 q$ for every $\frac{1}{\beta} < q < \frac{1}{1-\beta}$. The claim follows. □

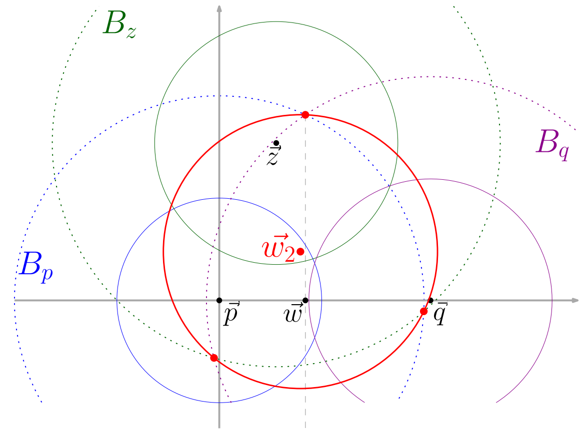
Remark 7. *A natural extension of the algorithm in Theorem 4 will be to allow another step. Specifically, to choose $\beta' > \beta$ so that Lemma 4 will not hold w.r.t. β'. Then, in case $\vec{w}$ is not a β'-plurality point, there is a point $\vec{z}$ such that the ball $B_{\mathbb{R}^d}(\vec{z}, \beta'\|\vec{z}-\vec{w}\|_2)$ contains more than $\frac{n}{2}$ voter points. Then, one might hope to find a new candidate point $\vec{w_2}$ that will be a β'-plurality point. Here a natural choice of $\vec{w_2}$ will be the center of the minimal ball containing the intersection of the three balls $B_p = B_{\mathbb{R}^d}(\vec{p}, 2)$, $B_q = B_{\mathbb{R}^d}(\vec{q}, \beta'\|\vec{q}-\vec{p}\|_2+1)$, and $B_z = B_{\mathbb{R}^d}(\vec{z}, \beta'\|\vec{z}-\vec{w}\|_2+1)$. See illustration below.*

Even though it is indeed possible that this approach will provide some improvement, it is unlikely to be significant. The reason is that even for the simplest symmetric case where $\vec{q} = (\frac{1}{\beta'}, 0)$, $\vec{z} = (\frac{1}{2\beta'}, \frac{1}{\beta'})$, one need $\beta' \leq \sqrt{\frac{89}{256}} \approx 0.59$. For the hardest case, it is likely that a much smaller β' will be required.

Missing Proofs

Equivalence Between the Definitions of β-Plurality

Lemma 8. *Definition 1 for $\beta(p, V)$ is equivalent to the definition from Aronov et al. (2020). Specifically, for a metric space (X, d), voter multiset V and point $p \in X$, denote by $\beta(p, V)$ the value defined in (Aronov et al. 2020), and by $\tilde{\beta}(p, V)$ the value from Definition 1. It holds that $\beta(p, V) = \tilde{\beta}(p, V)$ (note that the equivalence of the parameters $\beta_{(X,d)}$ and $\beta^*_{(X,d)}$ follows).*



Proof. Fix $|V| = n$. There are two directions for the proof:

- $\beta(p, V) \leq \tilde{\beta}(p, V)$. Assume by contradiction that $\tilde{\beta}(p, V) < \beta(p, V)$. By the definition of $\beta(p, V)$, there exists α, $\tilde{\beta}(p, V) < \alpha \leq \beta(p, V)$ such that for every $q \in X$, $|\{v \in V \mid \alpha\cdot d(p, v) < d(q, v)\}| \geq |\{v \in V \mid \alpha\cdot d(p, v) > d(q, v)\}|$, implying $|\{v \mid \alpha\cdot d(p, v) \leq d(q, v)\}| \geq \frac{n}{2}$. Thus $\tilde{\beta}(p, V) \geq \alpha$, a contradiction.
- $\tilde{\beta}(p, V) \leq \beta(p, V)$. Assume by contradiction that $\beta(p, V) < \tilde{\beta}(p, V)$, and let $\epsilon > 0$ such that $\beta(p, V) + \epsilon < \tilde{\beta}(p, V)$. By definition of $\tilde{\beta}(p, V)$, there is an $\alpha \geq \beta(p, V) + \epsilon$ such that for every q, $|\{v \in V \mid \alpha\cdot d(p, v) \leq d(q, v)\}| \geq \frac{n}{2}$. Let $\alpha' = \alpha - \frac{\epsilon}{2} \in (\beta(p, V), \alpha)$. Then for every $q \neq p$, $|\{v \in V \mid \alpha'\cdot d(p, v) < d(q, v)\}| \geq |\{v \in V \mid \alpha\cdot d(p, v) \leq d(q, v)\}| \geq \frac{n}{2}$, implying $|\{v \in V \mid \alpha'\cdot d(p, v) < d(q, v)\}| \geq |\{v \in V \mid \alpha'\cdot d(p, v) > d(q, v)\}|$. Clearly, for $q = p$, $|\{v \in V \mid \alpha'\cdot d(p, v) < d(q, v)\}| \geq |\{v \in V \mid \alpha'\cdot d(p, v) > d(q, v)\}|$. It follows that p is an α'-plurality point, a contradiction.

□

Proof of Theorem 3

We begin by defining the continuous counterpart $\tilde{G}$ of a weighted graph $G = (V, E, w)$. Each edge $e = (v, u)$ in G is represented in $\tilde{G}$ by a an interval of length $w(e)$ equipped with the line metric with endpoints u, v. $d_{\tilde{G}}$ is then the natural induced metric. That is, the distance between two points $u, v \in \tilde{G}$ denoted $d_{\tilde{G}}(u, v)$, is the shortest length of a geodesic path connecting u to v. We restate Theorem 3:

Theorem 3. *For every weighted graph $G = (V, E, w)$ containing a cycle, it holds that $\beta^*_{(\tilde{G}, d_{\tilde{G}})} \leq \frac{1}{2}$.*

Proof. Let C be a cycle in G of shortest length. Assume w.l.o.g. that the length of C is 3. We place 3 voters v_1, v_2, v_3 on $\tilde{C}$ at unit distance from each other. Assume by contradiction that there is a choice p which is a β-plurality point for $\beta > \frac{1}{2}$. Farther, assume w.l.o.g. that v_1 is the voter closest to p, and denote $d_{\tilde{G}}(p, v_1) = \alpha$.

If p lies on the cycle $\tilde{C}$, then the same argument as in Theorem 2 will imply a contradiction to the assumption that p is a β-plurality point.

Else, if v_1 lies on the shortest paths from p to both v_2, v_3, then $d_{\tilde{G}}(p, v_2), d_{\tilde{G}}(p, v_3) \geq 1$. Consider the choice q lying at distance $\frac{1}{2}$ from both v_2, v_3. Then as $\max\{d_{\tilde{G}}(q, v_2), d_{\tilde{G}}(q, v_3)\} = \frac{1}{2} < \beta \cdot \min\{d_{\tilde{G}}(p, v_2), d_{\tilde{G}}(p, v_3)\}$, q will win two votes over p, a contradiction.

Else, suppose w.l.o.g. that the shortest path from p to v_2 does not pass though v_1. Necessarily $d_G(p, v_2) \geq 2 - \alpha$, as otherwise v_1, p, v_2 will be a cycle in $\tilde{G}$ of length strictly less than 3, a contradiction to the minimality of C. Let q be the point lying on the shortest path from v_1 to v_2 at distance $\frac{\alpha}{2}$ from v_1 and $1 - \frac{\alpha}{2}$ from v_2. Note that q wins both the votes of v_1 and v_2 over p, a contradiction. □

Proof of Claim 2

Proof. By translation and rotation, we can assume w.l.o.g. that $\vec{a} = \vec{0}$, and $\vec{b} = \|\vec{a} - \vec{b}\|_2 \cdot e_1$ (e_1 here is the first standard basis vector). A straightforward calculation shows that

$$\begin{aligned}
&\left\{\vec{x} \in \mathbb{R}^d \mid \beta \cdot \|\vec{a} - \vec{x}\|_2 > \|\vec{b} - \vec{x}\|_2\right\} \\
&= \Bigg\{\vec{x} \in \mathbb{R}^d \mid \left(x_1 - \|\vec{a} - \vec{b}\|_2\right)^2 \\
&\qquad\qquad + \sum_{i=2}^{d} x_i^2 < \beta^2 \cdot \sum_{i=1}^{d} x_i^2\Bigg\} \\
&= \Bigg\{\vec{x} \in \mathbb{R}^d \mid \left(1 - \beta^2\right) x_1^2 - 2x_1\|\vec{a} - \vec{b}\|_2 + \|\vec{a} - \vec{b}\|_2^2 \\
&\qquad\qquad + \left(1 - \beta^2\right) \sum_{i=2}^{d} x_i^2 < 0\Bigg\} \\
&= \Bigg\{\vec{x} \in \mathbb{R}^d \mid \left(x_1 - \frac{\|\vec{a} - \vec{b}\|_2}{1 - \beta^2}\right)^2 \\
&\qquad\qquad + \sum_{i=2}^{d} x_i^2 < \frac{\beta^2\|\vec{a} - \vec{b}\|_2^2}{\left(1 - \beta^2\right)^2}\Bigg\} .
\end{aligned}$$

Thus we indeed obtain a ball with center at $\vec{o} = \frac{\|\vec{a}-\vec{b}\|_2}{1-\beta^2} \cdot e_1 = \vec{a} + \frac{1}{1-\beta^2} \cdot (\vec{a} - \vec{b})$, and radius $r = \sqrt{\frac{\beta^2\|\vec{a}-\vec{b}\|_2^2}{(1-\beta^2)^2}} = \beta \cdot \|\vec{o} - \vec{a}\|_2$. □

Proof of Equation (2)

Set

$$\begin{aligned}
f(\beta, q) &= \|\vec{i} - \vec{w}\|_2^2 = 4 - w^2 \\
&= 4 - \left(\frac{1}{2}(1 - \beta^2)q - \beta + \frac{3}{2q}\right)^2 .
\end{aligned}$$

We show that for our choice of β, $\forall q \in (\frac{1}{\beta}, \frac{1}{1-\beta})$, it holds that $\sqrt{f(\beta, q)} \leq \frac{1}{\beta}$, thus proving equation (2).

$$\frac{\partial}{\partial q} f(\beta, q) = 2\left(\frac{1}{2}(1 - \beta^2)q - \beta + \frac{3}{2q}\right)\left(\frac{1}{2}(1 - \beta^2) - \frac{3}{2q^2}\right)$$

which equals to 0 only for $q \in \left\{\pm\sqrt{\frac{3}{1-\beta^2}}, \frac{\sqrt{4\beta^2-3}\pm\beta}{\beta^2-1}\right\}$.

As we restrict our attention to $q \in (\frac{1}{\beta}, \frac{1}{1-\beta})$, it follows that once we fixed β, $f(\beta, q)$ has a maximum at $\sqrt{\frac{3}{1-\beta^2}}$ (note that $\sqrt{\frac{3}{1-\beta^2}} \in (\frac{1}{b}, \frac{1}{1-b})$ for every $b \in (\frac{1}{2}, 1)$). It thus will be enough to prove that

$$\begin{aligned}
f(\beta, q) &\leq f\left(\beta, \sqrt{\frac{3}{1 - \beta^2}}\right) \\
&= 1 + 2\beta^2 + 2\sqrt{3}\sqrt{1 - \beta^2}\beta \leq \frac{1}{\beta^2} .
\end{aligned}$$

This expression could be "massaged" into a degree 4 polynomial. Thus we can obtain an exact solution. In particular, for every $\beta \in \left(0, \frac{1}{2}\sqrt{\frac{1}{2} + \sqrt{3} - \frac{1}{2}\sqrt{4\sqrt{3} - 3}}\right] \approx (0, 0.557]$, it holds that $\sqrt{f(\beta, q)} \leq \frac{1}{\beta}$, as required.

Conclusion

Denote $\beta^* = \inf\left\{\beta^*_{(X,d)} \mid (X, d) \text{ is a metric space}\right\}$. In this paper we showed that $\sqrt{2} - 1 \leq \beta^* \leq \frac{1}{2}$. Further, in the Euclidean case, for arbitrary dimension $d \geq 4$, by combining our results with (Aronov et al. 2020), we know that $0.557 < \beta^*_{(\mathbb{R}^d, \|\cdot\|_2)} \leq \frac{\sqrt{3}}{2}$. The main question left open is closing these two gaps. Our conjecture is that the upper bounds are tight, since when $|V| = 3$, a plurality point must "win" $\frac{2}{3}$ of the overall vote. This task can only become easier once the number of voters increases.

Conjecture 1. *$\beta^* = \frac{1}{2}$, and for every $d \geq 2$,*

$$\beta^*_{(\mathbb{R}^d, \|\cdot\|_2)} = \frac{\sqrt{3}}{2} .$$

If indeed $\beta^*_{(\mathbb{R}^d, \|\cdot\|_2)} = \frac{\sqrt{3}}{2} \approx 0.866$ for every dimension d, then it implies that the concept of β-plurality might be very useful as a relaxation for Condorcet winner. Informally, it shows that the amount of "compromise" that we need to make in order to find a plurality point in any Euclidean space is relatively small.

Acknowledgments

After sharing our proof of Theorem 1 with the authors of (Aronov et al. 2020), Mark de Berg proved a weaker version of Theorem 4, and generously allowed us to publish our proof which is based on his observation. Specifically, de Berg proved that $\beta^*_{(\mathbb{R}^d,\|\cdot\|_2)} \geq \frac{1}{2}$.[4]

The authors would also like to thank Boris Aronov and Nimrod Talmon for providing useful comments on the manuscript.

Funding. Work by Arnold Filtser was supported by the Simons Foundation. Work by Omrit Filtser was supported by the Eric and Wendy Schmidt Fund for Strategic Innovation, by the Council for Higher Education of Israel, and by Ben-Gurion University of the Negev.

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[4] Following the lines of the proof of Theorem 4, for $\beta = \frac{1}{2}$, $\vec{p}$ is a $\frac{1}{2}$-plurality point, as no q satisfies equation (1).