

Engineering an Exact Pseudo-Boolean Model Counter*

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Abstract

Model counting, a fundamental task in computer science, involves determining the number of satisfying assignments to a Boolean formula, typically represented in conjunctive normal form (CNF). While model counting for CNF formulas has received extensive attention with a broad range of applications, the study of model counting for Pseudo-Boolean (PB) formulas has been relatively overlooked. Pseudo-Boolean formulas, being more succinct than propositional Boolean formulas, offer greater flexibility in representing real-world problems. Consequently, there is a crucial need to investigate efficient techniques for model counting for PB formulas.

In this work, we propose the first exact Pseudo-Boolean model counter, PBCount, that relies on knowledge compilation approach via algebraic decision diagrams. Our extensive empirical evaluation shows that PBCount can compute counts for 1513 instances while the current state-of-the-art approach could only handle 1013 instances. Our work opens up several avenues for future work in the context of model counting for PB formulas, such as the development of preprocessing techniques and exploration of approaches other than knowledge compilation.

1 Introduction

Propositional model counting involves computing the number of satisfying assignments to a Boolean formula. Model counting is closely related to the Boolean satisfiability problem where the task is to determine if there exists an assignment of variables such that the Boolean formula evaluates to *true*. Boolean satisfiability and model counting have been extensively studied in the past decades and are the cornerstone of an extensive range of real-life applications such as software design, explainable machine learning, planning, and probabilistic reasoning (Bacchus, Dalmao, and Pitassi 2003; Narodytska et al. 2019; Jackson 2019; Fan, Miller, and Mitra 2020). Owing to decades of research, there are numerous tools and techniques developed for various aspects of Boolean satisfiability and model counting, from Boolean formula preprocessors to SAT solvers and model counters.

The dominant representation format of Boolean formulas is Conjunctive Normal Form (CNF), and accordingly, the tools in the early days focused on CNF as the input format. Over the past decade and a half, there has been considerable effort in exploring other representation formats: one such format that has gained significant interest from the community is Pseudo-Boolean (PB) formulas, which are expressed as the conjunction of linear inequalities. PB formulas are shown to be more succinct than CNF formulas and natural for problems such as Knapsack, sensor placement, binarized neural networks, and the like. Furthermore, PB formulas are able to express constraints more succinctly compared to Boolean formulas in CNF (Berre et al. 2018). As an example, a single PB constraint is sufficient to express at-most- k and at-least- k types of cardinal constraints whereas the equivalent in CNF would require a polynomial number of clauses (Sinz 2005). On a higher level, an arbitrary CNF clause can be expressed with a single PB constraint but the converse is not true (Berre et al. 2018). The past decade has witnessed the development of satisfiability solving techniques based on the underlying proof systems naturally suited to PB constraints, and accordingly, the state-of-the-art PB solvers, such as RoundingSat significantly outperform CNF solvers on problems that are naturally encoded in PB (Elffers and Nordström 2018; Devriendt 2020; Devriendt et al. 2021).

In contrast to satisfiability, almost all the work in the context of model counting has focused on the representation of Boolean formulas in Conjunctive Normal Form (CNF), with the sole exception of the development of an approximate model counter for PB formulas (Yang and Meel 2021).

The primary contribution of this work is to address the aforementioned gap through the development of a native scalable exact model counter, called PBCount, for PB formulas. PBCount is based on the knowledge compilation paradigm, and in particular, compiles a given PB formula into algebraic decision diagrams (ADDs) (Bahar et al. 1993), which allows us to perform model counting. We perform extensive empirical evaluations on benchmark instances arising from different applications, such as sensor placement, multi-dimension knapsack, and combinatorial auction benchmarks (Gens and Levner 1980; Blumrosen and Nisan 2007; Latour, Sen, and Meel 2023). Our evaluations highlighted the efficacy of PBCount against ex-

*The full version of the paper is available at <https://arxiv.org/abs/2312.12341> and code is available at <https://github.com/grab/pbcount>
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isting state-of-the-art CNF model counters. In particular, PBCount is able to successfully count 1513 instances while the prior state of the art could only count 1013 instances, thereby demonstrating significant runtime improvements. It is worth remarking that PBCount achieves superior performance with substantially weaker preprocessing techniques in comparison to techniques employed in CNF model counters, making a strong case for the advantages of native PB model counting and reasoning. Furthermore, given the crucial importance of preprocessing techniques for CNF counting, we hope our work will motivate the development of preprocessing techniques for PB model counting.

The rest of the paper is organized as follows: We discuss the preliminaries and existing counting algorithm in Section 2. In Section 3, we discuss existing works and how they relate to our approach, which we detail in Section 4. Following that, we analyze the empirical results of PBCount against existing tools in Section 5 and conclude in Section 6.

2 Preliminaries

Boolean Formula A Boolean variable can take values *true* or *false*. A literal is either a Boolean variable or its negation. Let F be a Boolean formula. F is in conjunctive normal form (CNF) if F is a conjunction of clauses, where each clause is a disjunction of literals. F is satisfiable if there exists an assignment τ of variables of F such that F evaluates to *true*. We refer to τ as a satisfying assignment of F and denote the set of all τ as $\text{Sol}(F)$. Model counting for Boolean formula F refers to the task of determining $|\text{Sol}(F)|$.

Pseudo-Boolean Formula A PB constraint is either an equality or inequality of the form $\sum_{i=1}^n a_i x_i \square k$ where x_1, \dots, x_n are Boolean literals, a_1, \dots, a_n , and k are integers, and \square is one of $\{\geq, =, \leq\}$. We refer to a_1, \dots, a_n as term coefficients in the PB constraint, where each term is of the form $a_i x_i$. A PB formula, G , consists of a set of PB constraints. G is satisfiable if there exists an assignment τ of all variables of G such that all its PB constraints hold. PB model counting refers to the computation of $|\text{Sol}(G)|$ where $\text{Sol}(G)$ is the set of all satisfying assignments of G .

Projected Model Counting Let G be a formula defined over the set of variables X . Let V_i, V_j be subsets of X such that $V_i \cap V_j = \emptyset$ and $V_i \cup V_j = X$. Projected model counting of G on V_i refers to the number of assignments of all variables in V_i such that there exists an assignment of variables in V_j that makes G evaluate to *true* (Aziz et al. 2015). In the evaluations, CNF model counter baselines perform projected model counting on the original variables in the PB formula, to avoid additional counts due to auxiliary variables introduced in the PB to CNF conversion process.

Algebraic Decision Diagram An algebraic decision diagram (ADD) is a directed acyclic graph representation of a function $f : 2^X \rightarrow S$ where X is the set of Boolean variables that f is defined over, and S is an arbitrary set known as the carrier set. We denote the function represented by an ADD ψ as $\text{Func}(\psi)$. The internal nodes of ADD represent decisions on variables $x \in X$ and the leaf nodes represent $s \in S$. In this work, we focus on the setting where $S \subset \mathbb{Z}$.

As an example, an ADD representing $3x_1 + 4x_2$ is shown in Figure 1. In the figure, a dotted arrow from an internal node represents when the corresponding variable is set to *false* and a solid arrow represents when it is set to *true*.

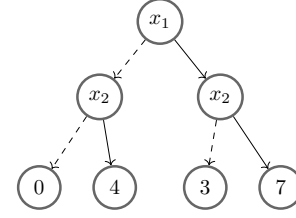


Figure 1: An ADD representing $3x_1 + 4x_2$

In addition, we make use of Apply and ITE operations on ADDs (Bryant 1986; Bahar et al. 1993). The Apply operation takes as input a binary operator \bowtie , two ADDs ψ_1, ψ_2 , and outputs an ADD ψ_3 such that $\text{Func}(\psi_3) = \text{Func}(\psi_1) \bowtie \text{Func}(\psi_2)$. The ITE operation (*if-then-else*) involves 3 ADDs ψ_1, ψ_2, ψ_3 , where carrier set of ψ_1 is restricted to $\{0, 1\}$. ITE outputs an ADD that is equivalent to having 1 valued leaf nodes in ψ_1 replaced with ψ_2 and 0 valued leaf nodes with ψ_3 .

Relation of Pseudo-Boolean Constraint to CNF Clause Given an arbitrary CNF clause D , one could always convert D to a PB constraint. Given that D is of the form $\bigvee_{i=1}^m l_i$, where l_1, \dots, l_m are Boolean literals, D can be represented by a single PB constraint $\sum_{i=1}^m a_i l_i \geq 1$ where all coefficients $a_1, \dots, a_i, \dots, a_m$ are 1. However, there are PB constraints that require polynomially many CNF clauses to represent. An example would be $\sum_{i=1}^m l_i \geq k$ which requires at least k of m literals to be *true*. We refer the reader to the Appendix for statistics of the number of variables and clauses before and after PB to CNF conversion for benchmarks used.

2.1 Model Counting with ADDs

In this work, we adapt the existing dynamic programming counting algorithm of ADDMC (Dudek, Phan, and Vardi 2020a), shown in Algorithm 1, to perform PB model counting with ADDs. This includes using the default ADDMC configurations for ADD variable ordering (MCS) and cluster ordering ρ (BOUQUET_TREE). The algorithm takes in a list φ of ADDs, representing all constraints, and an order ρ of which to process the ADDs. ADD ψ is initialized with value 1. According to cluster ordering ρ , cluster ADDs ψ_j are formed using the Apply operation with \times operator on each of the individual constraint ADDs of constraints in the cluster. The cluster ADD ψ_j is combined with ψ using the same Apply operation. If variable x does not appear in later clusters in ρ , it is abstracted out from ψ (early projection process in ADDMC) using $\psi \leftarrow \mathcal{W}(\bar{x}) \times \psi[x \mapsto 0] + \mathcal{W}(x) \times \psi[x \mapsto 1]$ in line 8, where $\mathcal{W}(\cdot)$ is the user provided literal weight function. In unweighted model counting, $\mathcal{W}(\cdot)$ is 1 for all literals. Once all clusters have been processed, the unprocessed variables x of the formula G are abstracted out using the same operation as before (line 10).

After all variables are abstracted out, ψ is a constant ADD that represents the final count.

Algorithm 1: computeCount(φ, ρ)

Input: φ - list of ADD, ρ - cluster merge ordering

Output: model count

```

1:  $\psi \leftarrow \text{constantADD}(1)$ 
2: for cluster  $A_j \in \rho$  do
3:    $\psi_j \leftarrow \text{constantADD}(1)$ 
4:   for constraint  $C_i \in A_j$  do
5:      $\psi_j \leftarrow \psi_j \times \varphi[C_i]$ 
6:    $\psi \leftarrow \psi \times \psi_j$ 
7:   for each  $x \in \psi$  where  $x$  not in later clusters in  $\rho$  do
8:      $\psi \leftarrow \mathcal{W}(\bar{x}) \times \psi[x \mapsto 0] + \mathcal{W}(x) \times \psi[x \mapsto 1]$ 
9:   for all unprocessed variable  $x$  do
10:     $\psi \leftarrow \mathcal{W}(\bar{x}) \times \psi[x \mapsto 0] + \mathcal{W}(x) \times \psi[x \mapsto 1]$ 
11: return getValue( $\psi$ )

```

3 Related Work

Boolean Formula Preprocessing Boolean formula preprocessing involves simplifying a given formula to reduce runtimes of downstream tasks such as determining satisfiability of the formula (SAT-solving) and model counting. Preprocessing is crucial to modern SAT solvers and model counters’ performance improvements in recent decades. There are numerous preprocessing techniques introduced over the years by the research community, some of which are *unit propagation*, *bounded variable elimination*, *failed literal probing*, and *vivification* (Dowling and Gallier 1984; Berre 2001; Eén and Biere 2005; Piette, Hamadi, and Sais 2008). In this work, we adapt some of the SAT preprocessing techniques, namely *unit propagation* and a variant of *failed literal probing*, to simplify PB formulas.

Search-Based Model Counters Among the numerous existing CNF model counters, we can classify them into two main categories – search-based model counters and decision diagram-based model counters. Notable existing search-based model counters include GPMC, Ganak, and Sharsat-TD (Ryosuke Suzuki and Sakai 2017; Sharma et al. 2019; Korhonen and Järvisalo 2021). Search-based model counters work by setting values to variables in a given formula in an iterative manner, which is equivalent to implicitly exploring a search tree. In addition, search-based model counters adapt techniques such as sub-component caching from SAT solving for more efficient computation.

Decision Diagram-Based Model Counter Decision diagram-based model counters employ knowledge compilation techniques to compile a given formula into directed acyclic graphs (DAGs) and perform model counting with these DAGs. Some of the recent decision diagram-based model counters are D4, ExactMC, ADDMC, and its related variant DPMC (Lagniez and Marquis 2017; Dudek, Phan, and Vardi 2020a,b; Lai, Meel, and Yap 2021). D4 and ExactMC compile the formula in a top-down manner into the respective decision diagram forms. In contrast, ADDMC and DPMC (decision diagram mode) perform bottom-up

compilations of algebraic decision diagrams (ADDs). In this work, we based PBCount on ADDMC and introduced techniques to compile a PB constraint directly into an ADD and employ the same counting approach in ADDMC.

Pseudo-Boolean Conversion One way to perform PB model counting is to convert the PB formula to a Boolean formula and use existing CNF model counters. A notable tool for the conversion of PB to CNF is PBLib (Philipp and Steinke 2015). PBLib implements various encodings to convert PB formulas into CNF form, some of which include cardinality networks, sorting networks, and BDD-based encodings (Eén and Sörensson 2006; Abío et al. 2011, 2013). In this work, we use default settings for the PBEncoder binary provided as part of PBLib to perform the required conversions. We subsequently compare PBCount against state-of-the-art CNF model counters. It is worth noting that the model counting task for PB formula becomes a projected model counting task of the corresponding CNF formula, as previously mentioned in Section 2.

4 Approach

We show the overall flow of PBCount in Figure 2. We first preprocess the PB formula using *propagation* and *assumption probing*. Subsequently, we compile each of the PB constraints into an algebraic decision diagram (ADD). Next, we merge constraint ADDs using Apply operation and perform model counting by abstracting out variables (Section 2.1). The model count would be the value after all variables are abstracted out. Without loss of generality, the algorithms described in this work handle PB constraints involving ‘=’ and ‘≥’ operators, as ‘≤’ type constraints can be manipulated into ‘≥’ type constraints.

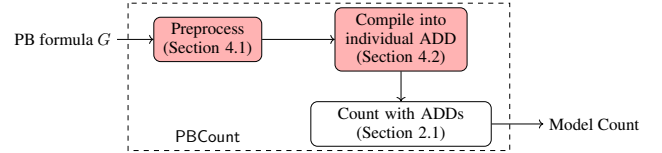


Figure 2: Overall flow of our PB model counter PBCount. Shaded boxes indicate our contributions.

4.1 Preprocessing

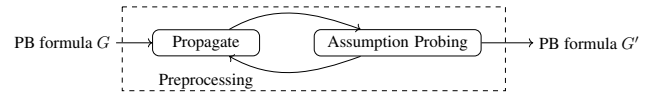


Figure 3: Preprocessing of PB formula

The preprocessing phase of PBCount performs assumption probing and unit propagation (Biere, Järvisalo, and Kiesl 2021). PBCount repeatedly performs unit propagation and assumption probing until no change is detected, as shown in Algorithm 2.

Algorithm 2: Preprocess(G)**Input:** G - PB formula**Output:** G' - preprocessed PB formula

```

1: mapping  $\leftarrow []$ ;  $G' \leftarrow G$ 
2: repeat
3:   for all single variable constraint  $C \in G'$  do
4:     mapping  $\leftarrow$  mapping  $\cup$  InferDecision( $C$ )
5:    $G' \leftarrow$  propagate( $G'$ , mapping)
6:   for all variable  $x \in G'$  do
7:     mapping  $\leftarrow$  mapping  $\cup$  AssumProbe( $G'$ ,  $x$ )
8:    $G' \leftarrow$  propagate( $G'$ , mapping)
9: until  $G'$  does not change
10: return  $G'$ 

```

Sign Manipulation Let C be the PB constraint $-3x_1 - 4x_2 \leq -3$. One can multiply both sides of the constraint by -1 to form $3x_1 + 4x_2 \geq 3$. In addition, one would be able to switch the sign of the coefficient of x_2 as follows.

$$\begin{aligned}
3x_1 + 4x_2 &\geq 3 \\
3x_1 + 4(1 - \bar{x}_2) &\geq 3 \\
3x_1 - 4\bar{x}_2 &\geq -1
\end{aligned}$$

In general, one is able to manipulate the sign of any term coefficient as shown in the example above. We use the above technique to optimize PB constraint compilation approaches which we discuss in later sections.

Propagation Propagation in the Pseudo-Boolean context refers to the simplification of the PB constraints if decisions on some PB variables can be inferred. In particular, one might be able to infer decisions on PB variable x_i from PB constraint C_j when the constraint is of either 1) $a_i x_i \geq k$ or 2) $a_i x_i = k$ forms. We defer the details of the InferDecision algorithm to the Appendix.

Algorithm 3: AssumProbe(G, x_i)**Input:** G - PB formula, x_i - assumption variable**Output:** mapping of variable values

```

1: temp, mapping  $\leftarrow []$ 
2: for all constraint  $C \in G[x_i \mapsto 1]$  do
3:   temp  $\leftarrow$  temp  $\cup$  InferDecision( $C$ )
4: for all constraint  $C \in G[x_i \mapsto 0]$  do
5:   temp  $\leftarrow$  temp  $\cup$  InferDecision( $C$ )
6: for all variable  $x_j$ , where  $j \neq i$  do
7:   if exactly one literal of  $x_j$  in temp then
8:     mapping  $\leftarrow$  mapping  $\cup$  temp[ $x_j$ ]
9: return mapping

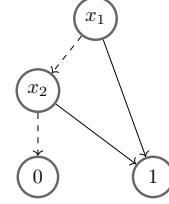
```

Assumption Probing Assumption probing can be viewed as a weaker form of failed literal probing (Biere, Jarvisalo, and Kiesl 2021) as well as single step look ahead propagation process. For an arbitrary variable $x_i \in G$, where G is the PB formula, assumption probing involves performing propagation and decision inference independently for when $x_i = 0$ and $x_i = 1$. If another variable x_j is inferred to have the same value assignment $\tau[x_j]$ in both cases, then it

can be inferred that x_j should be set to $\tau[x_j]$ in all satisfying assignments of G . Algorithm 3 illustrates the process for a single variable x_i , and in the preprocessing stage, we perform assumption probing on all variables in G .

4.2 Pseudo-Boolean Constraint Compilation

In this work, we introduce two approaches, namely top-down and bottom-up, to compile each constraint of a PB formula into an ADD. We use T, k , and eq in place of PB constraint C when describing the compilation algorithms. T refers to the term list, which is a list of $a_i x_i$ terms of C . k is the constraint constant and eq indicates if C is '=' constraint.

Figure 4: An ADD ψ_1 representing $3x_1 + 4x_2 \geq 3$

Bottom-up ADD Constraint Compilation In order to compile an ADD which represents a PB constraint of the following form $\sum_{i=1}^n a_i x_i [\geq, =, \leq] k$, we first start compiling the expression $\sum_{i=1}^n a_i x_i$ from literal and constant ADDs as shown by line 3 of Algorithm 4. A constant ADD which represents integer a_i is a single leaf node that has value a_i . A literal ADD comprises of an internal node, which represents variable x , and true and false leaf nodes, which represent the evaluated values of the literal if x is set to true and false. With the literal and constant ADDs, we use Apply with \times operator to form ADDs for each term $a_i x_i$. We use Apply with $+$ operator on term ADDs to form the ADD representing expression $\sum_{i=1}^n a_i x_i$. As an example, the ADD ψ for the expression $3x_1 + 4x_2$ is shown in Figure 1. To account for the inequality or equality, we look at the value of leaf nodes in expression ADD ψ and determine if they satisfy the constraint (lines 4 to 10). We replace the leaf nodes with 1 node if the constraint is satisfied and 0 node otherwise, the resultant ADD is illustrated in Figure 4.

Top-down ADD Constraint Compilation In contrast to the bottom-up ADD compilation approach, the top-down ADD compilation for a given PB constraint involves the if-then-else (ITE) operation for decision diagrams. We only consider PB constraints that involve $=$ or \geq as mentioned previously. The top-down compilation algorithm (Algorithm 5) makes use of recursive calls of Algorithm 6 to construct an ADD that represents a given PB clause. In particular, Algorithms 5 and 6 work by iterating through the terms of the PB constraint using idx . The algorithms build the sub-ADDs when the literal at position idx evaluates to *true* for the if-then case and otherwise for the else case of the ITE operation while updating the constraint constant k (lines 2-3 of Algorithm 5 and lines 9-10 of Algorithm 6). Notice that the top-down compilation approach allows for early termination when the current k value is negative for $\geq k$ case. However,

Algorithm 4: compileConstraintBottomUp(T, k, eq)

Input: T - term list, k - constraint value,
 eq - indicator if constraint is '=' type
Output: ψ - constraint ADD

```

1:  $\psi \leftarrow \text{constantADD}(0)$ 
2: for term  $t$  in  $T$  do
3:    $\psi \leftarrow \text{constantADD}(t.\text{coeff}) \times \text{literalADD}(t.\text{literal})$ 
4: for node  $n$  in  $\text{LeafNode}(\psi)$  do
5:   if  $eq$  is true &  $n.\text{value} = k$  then
6:      $n.\text{value} \leftarrow 1$ 
7:   else if  $eq$  is false &  $n.\text{value} \geq k$  then
8:      $n.\text{value} \leftarrow 1$ 
9:   else
10:     $n.\text{value} \leftarrow 0$ 
11: return  $\psi$ 

```

early termination is possible only if all unprocessed coefficients are positive, implying that k in subsequent recursive calls cannot increase. One way would be to sort the term list T in ascending order of term coefficients, processing terms with negative coefficients before positive coefficients.

Algorithm 5: compileConstraintTopDown(T, k, eq)

Assumption: T is in ascending order of term coefficients or all coefficients are non-negative
Input: T - term list, k - constraint value,
 eq - indicator if constraint is '=' type
Output: ψ - constraint ADD

```

1:  $\psi \leftarrow \text{literalADD}(T[0].\text{literal})$ 
2:  $\psi_{lo} \leftarrow \text{compileTDRecur}(T, k, eq, 1)$ 
3:  $\psi_{hi} \leftarrow \text{compileTDRecur}(T, k - T[0].\text{coeff}, eq, 1)$ 
4:  $\psi.\text{ITE}(\psi_{hi}, \psi_{lo})$ 
5: return  $\psi$ 

```

Optimizations for Bottom-up Compilation In the bottom-up compilation approach, an ADD is built from the individual literal and constant ADDs to represent the expression, before subsequently having leaf node values converted to 1 and 0 depending on if the PB constraint is satisfied. In the process, an ADD could be exponential in size with respect to the number of variables processed. In order to minimize the intermediate ADD during the compilation process, we introduce an optimization for bottom-up compilation. The key idea is to increase the number of shared sub-components of the intermediate ADD, and this amounts to processing the PB constraint terms in a manner that results in fewer distinct subset sums of term coefficients as every distinct subset sum requires a separate leaf node. To this end, we optimize the compilation process by sorting the terms according to the absolute values of their coefficients in ascending order. Subsequently, we manipulate the coefficients, using $x = (1 - \bar{x})$, of the terms such that alternate terms have coefficients of different signs. We defer the pseudo code to the Appendix.

Algorithm 6: compileTDRecur(T, k, eq, idx)

Input: T - term list, k - current constraint value,
 eq -input constraint equality, idx -index of current term in T
Output: ψ - constraint ADD from idx to end of T

```

1: if  $T[idx].\text{coeff} \geq 0$  then
2:    $\text{isPos} \leftarrow \text{true}$ 
3:   if  $eq$  &  $\text{isPos}$  &  $k < 0$  then
4:     return  $\text{constantADD}(0)$ 
5:   else if  $!eq$  &  $\text{isPos}$  &  $k \leq 0$  then
6:     return  $\text{constantADD}(1)$ 
7:   else if  $idx < T.\text{length}$  then
8:      $\psi \leftarrow \text{literalADD}(T[idx].\text{literal})$ 
9:      $\psi_{lo} \leftarrow \text{compileTDRecur}(T, k, eq, idx + 1)$ 
10:     $\psi_{hi} \leftarrow \text{compileTDRecur}(T, k - T[idx].\text{coeff}, eq, idx + 1)$ 
11:    return  $\psi.\text{ITE}(\psi_{hi}, \psi_{lo})$ 
12: else
13:   if  $eq$  &  $k = 0$  then
14:     return  $\text{constantADD}(1)$ 
15:   else
16:     return  $\text{constantADD}(0)$ 

```

Optimizations for Top-down Compilation Similarly, we also introduce optimizations for the top-down compilation approach. Recall that one would only be able to perform early termination for PB constraints of the form $\sum a_i x_i \geq k$ after all negative coefficient terms have been processed. To this end, we manipulate all coefficients to be positive and adjust k accordingly so that early termination is possible. Furthermore, we sort the terms in descending value of the term coefficients as larger coefficients are more likely to satisfy the constraint. We defer the pseudo code to the Appendix.

Algorithm 7: compileConstraintDynamic(T, k, eq)

Input: T - term list, k - constraint value, eq -input constraint equality
Output: ψ - constraint ADD

Cond 1: $T.\text{length} \leq 25$ **and** $k < 25^{\text{th}}$ percentile of $T.\text{coeff}$
Cond 2: $k < 25^{\text{th}}$ percentile of $T.\text{coeff}$ **and** unique coefficient rate ≥ 0.9 **and** unique adjacent difference rate ≥ 0.85

```

1: if cond 1 or cond 2 then
2:    $\text{bottomUp} \leftarrow \text{false}$ 
3: else
4:    $\text{bottomUp} \leftarrow \text{true}$ 
5: if  $\text{bottomUp}$  then
6:   return  $\text{optimizeCompileBottomUp}(T, k, eq)$ 
7: else
8:   return  $\text{optimizeCompileTopDown}(T, k, eq)$ 

```

Dynamic Compilation A PB formula can include more than one PB constraint. As we will show in a case study in the experiments section, the choice of compilation approach has a substantial impact on overall runtime. To this end, we introduce a dynamic heuristic (Algorithm 7) to select the appropriate compilation approach and perform optimization of the compilation process as previously discussed. In Algo-

rithm 7, we choose top-down compilation if either condition 1 or 2 is met. Conditions 1 and 2 are designed to be in favor of the bottom-up compilation approach, we provide performance analysis in the experiments section.

5 Experiments

We performed extensive empirical evaluations to compare the runtime performance of PBCount with state-of-the-art exact model counters. Our empirical evaluation focuses on benchmarks arising from three application domains: sensor placement, auctions, and multi-dimensional knapsack. Through our evaluations and analysis, we sought to answer the following research questions:

RQ 1 How does the runtime performance of PBCount compare to that of the state-of-the-art approaches?

RQ 2 How does the dynamic compilation approach impact the runtime performance of PBCount?

Setup We performed our evaluations on machines with AMD EPYC 7713 processors. Each benchmark instance is provided with 1 core, 16GB memory, and a timeout of 3600 seconds. Since all the state-of-the-art exact model counters take CNF as input, we employed the CNF model counters with the help of PB to CNF conversion tool PBLib¹ (Philipp and Steinke 2015). We evaluated PBCount against state-of-the-art projected counters: DPMC, D4² and GPMC; D4 and GPMC are among the winners of the Projected counting track at Model Counting Competition 2022 and 2023.

Benchmarks We generated 3473 benchmarks of the following application areas – sensor placement, auctions, and multi-dimension knapsack. We detail the benchmark statistics (number of variables and constraints) in the Appendix.

- The sensor placement benchmark setting (1473 instances after removal of 0 counts) is adapted from prior work on identifying code sets (Latour, Sen, and Meel 2023). Given a network graph, a maximum number of sensors allowed, count the number of ways to place sensors such that failures in the network are uniquely identifiable.
- For the auction benchmark setting (1000 instances), we adapt the combinatorial auction setting (Blumrosen and Nisan 2007) to a counting variant. There are m participants and n items, each of which can be shared by one or more participants. Given that each participant has a minimum utility threshold, we count the number of ways the n items can be shared such that all participants achieve their minimum threshold. The utilities are additive and can be negative.
- For the multi-dimension knapsack benchmark setting (Gens and Levner 1980) (1000 instances), there are n items and constraints on m different features or dimensions of the items in the form of the sum of each dimension should not exceed a given constant. Given such a setting, the goal would be the count the number of sub-sets of items that satisfies the constraints.

¹We used the provided PBEncoder for conversion.

²Binary from Model Counting Competition 2022

5.1 RQ1: Runtime Comparison

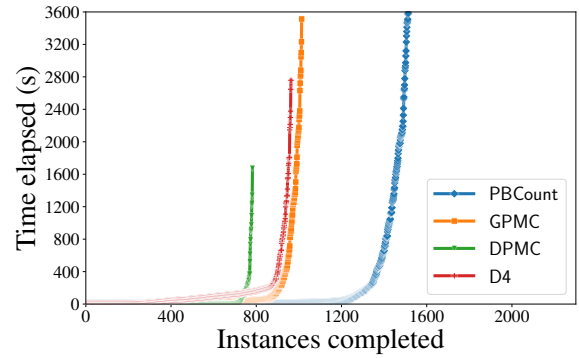


Figure 5: Cactus plot of number of benchmark instances completed by different counters. A point (x, y) on each line plot indicates the corresponding counter completes x number of benchmarks after y seconds has elapsed.

We show the cactus plot of the number of instances completed by each counter out of the 3473 benchmarks in Figure 5. The exact number of instances completed by each counter for each benchmark set is shown in Table 1. Additionally, we provide individual cactus plots for each set of benchmarks in the Appendix.

Benchmarks	DPMC	D4	GPMC	PBCount
Sensor placement	625	566	575	638
\mathcal{M} -dim knapsack	81	281	279	503
Auction	76	116	159	372
Total	782	963	1013	1513

Table 1: Number of benchmark instances completed by each counter in 3600s, higher is better.

In sensor placement benchmarks, PBCount count completed 638 instances, narrowly ahead of DPMC (625 instances), and more than D4 (566 instances) and GPMC (575 instances). In multi-dimension knapsack (\mathcal{M} -dim knapsack) and auction benchmarks, PBCount significantly outperforms the competing counters. PBCount completed 503 \mathcal{M} -dim knapsack instances, around $1.8\times$ that of GPMC (279 instances) and D4 (281 instances), and $6.2\times$ that of DPMC (81 instances). In auction benchmarks, PBCount completed 372 instances, around $2.3\times$ that of GPMC (159 instances), $3.2\times$ of D4 (116 instances), and $4.9\times$ of DPMC (76 instances). Overall, PBCount completed 1513 instances out of 3473 total instances, around $1.5\times$ that of GPMC, $1.6\times$ of D4, and $1.9\times$ that of DPMC. Note that PBCount achieved superior performance with minimal preprocessing over GPMC, which has advanced preprocessing capabilities. Our results demonstrate the significant performance advantages of counting natively for PB formulas and provide an affirmative answer to **RQ1**.

5.2 RQ2: Analysis of Compilation Approaches

We now focus on the analysis of different compilation approaches: top-down (Algorithm 5), bottom-up (Algorithm 4), and dynamic (Algorithm 7). The results in Table 2 show that for the benchmarks, bottom-up PB constraint compilation outperforms top-down approach significantly in auction and multi-dimension knapsack and to a lesser degree sensor placement. In addition, the evaluation result also highlights that our dynamic compilation heuristic and constraint term optimization closely match the bottom-up approach, with the exception of completing 3 fewer instances in auction benchmarks. However, in the 372 auction instances completed by both bottom-up and dynamic approaches, the dynamic approach with term coefficient optimization completes the counting task faster for 257 instances. We show the scatter plot comparison in Figure 6.

Benchmarks	Top-down	Bottom-up	Dynamic
Sensor placement	580	638	638
\mathcal{M} -dim knapsack	109	503	503
Auction	158	375	372

Table 2: Number of benchmarks completed by PBCount when employing different compilation strategies, higher number indicates better performance.

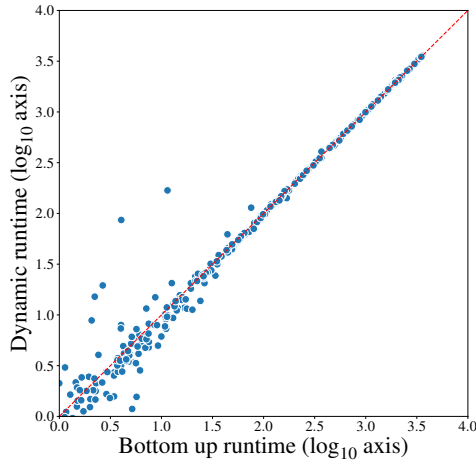


Figure 6: Dynamic vs bottom-up runtime (\log_{10}) for auction benchmarks. Points beneath red diagonal line indicate dynamic compilation is faster (257 points), points above otherwise (115 points).

Compilation Approach Performance Case Study We provide an example to highlight the performance impact of the choice of compilation approach. The example involves the following PB formula in Equation 1 with a single constraint that has unique term coefficients:

$$\sum_{i=0}^{12} 2^i x_{i+1} + \sum_{i=1}^{10} 3^i x_{i+13} + \sum_{i=1}^7 7^i x_{i+23} \geq k \quad (1)$$

We vary the value of k in the above PB constraint from 10^1 to 10^5 and compare the runtime between top-down and bottom-up compilation approaches in Table 3. Note that bottom-up compilation takes around the same time irrespective of k as there is no early termination. On the other hand for top-down compilation, the PB constraint is easily satisfied when k is small and thus allows for early termination, leading to significant time savings compared to when k is large. Notice that when top-down compilation is unable to terminate early, it is much slower than bottom-up compilation even when all term coefficients are unique.

Approach	k value				
	10^1	10^2	10^3	10^4	10^5
Top-down	0.005	0.009	0.228	8.586	46.071
Bottom-up	6.927	7.202	7.198	7.434	6.732

Table 3: Runtime (seconds) to complete model counting for formula in Equation 1. Lower is better

Approach	k value				
	10^1	10^2	10^3	10^4	10^5
Top-down	3.325	61.753	60.530	60.881	64.097
Bottom-up	0.005	0.004	0.004	0.004	0.004

Table 4: Runtime (seconds) to complete model counting for formula in Equation 1 with all coefficients set to 1.

As mentioned previously, bottom-up compilation benefits from having large numbers of same term coefficients or collisions in subset sums of coefficients. To this end, we changed all term coefficients of the PB constraint in equation 1 to 1 and compared runtimes in Table 4. We observed around three orders of magnitude reduction in the runtime of the bottom-up compilation approach. In contrast, the top-down approach terminates early only in $k = 10^1$ case and requires full enumeration in other cases. In the absence of early termination, top-down compilation approach is much slower than bottom-up compilation approach, and this is reflected in our dynamic compilation heuristic.

6 Conclusion

In this work, we introduce the first exact PB model counter, PBCount. PBCount directly compiles PB formulas into ADDs, enabling us to reuse the ADD counting framework in ADDMC. In the design of PBCount, we introduce both top-down and bottom-up PB constraint compilation techniques and highlight the performance differences between them. While we introduced dynamic compilation heuristics to determine the per constraint compilation method and preliminary preprocessing techniques for PB formulas, it would be of interest to develop more advanced heuristics and preprocessing techniques in future works. A strong motivation is PBCount’s performance lead over existing CNF model counters. We hope this work will gather more interest in PB formulas and PB model counting.

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