Minimal Change in Modal Logic S5

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Abstract

We extend belief revision theory from propositional logic to the modal logic S5. Our first contribution takes the form of three new postulates (M1-M3) that go beyond the AGM ones and capture the idea of minimal change in the presence of modalities. Concerning the construction of modal revision operations, we work with set pseudo-distances, i.e., distances between sets of points that may violate the triangle-inequality. Our second contribution is the identification of three axioms (A3-A5) that go beyond the standard axioms of metrics. Loosely speaking, our main result states the following: if a pseudo-distance satisfies certain axioms, then the induced revision operation satisfies (M1-M3). We investigate three pseudo-distances from the literature (Dhaus, Dinj, Dsum), and the three induced revision operations ($*_{\mathsf{Dhaus}}$, $*_{\mathsf{Dinj}}$, $*_{\mathsf{Dsum}}$). Using our main result, we show that only $*_{\mathsf{Dsum}}$ satisfies (M1-M3) all together. As a last contribution, we revisit a major criticism of AGM operations, namely that the revisions of $p \land q$ and $p \land (p \to q)$ are identical. We show that the problem disappears if instead of material implication we use the modal operator of strict implication that can be defined in S5.

1 Introduction

Logical languages with modalities provide expressive power beyond propositional logic: they allow us to distinguish between necessary truths $p \land \Box p$ and contingent truths $p \land \neg\Box p$. This distinction accounts for things such as laws of nature and database integrity constraints. The belief revision literature in the AGM tradition (Alchourrón, Gärdenfors, and Makinson 1985) has neglected the problem of how to revise belief sets containing modalities: as the SEP entry on belief revision says, "it is fair to say that operations of AGM-style belief change have not yet been constructed that are generally recognized as able to deal adequately with conditional or modal sentences" (Hansson 2022). We address this shortcoming by providing an axiomatic and semantic analysis of belief revision operations for modal logic.

On the axiomatic side, we formulate three simple rationality postulates and argue why a reasonable revision operation should satisfy them. On the semantic side, the main difficulty is to lift the orderings (on which the semantics of revision operations are based) from propositional logic valuations to Kripke models. For a start, we choose the modal logic S5 because it has an alternative, simpler semantics in terms of sets of valuations. We propose a family of distance-based revision operations. Each of them is based on a lifting of the Hamming distance from valuations to sets of valuations. The most interesting one sums up the minimal distances from each valuation of one set to the other set: we show that it validates all postulates.

Our revision operations allow us to shed new light on an old debate in belief revision: whether revision operations should be syntax-sensitive or not. According to the AGM postulate of syntax insensitivity, when two logically equivalent belief bases are revised by the same new piece of information then the two outcomes should also be logically equivalent. The main argument against syntax insensitivity is that while the two belief bases $\{p, q\}$ and $\{p, p \to q\}$ are 'statically equivalent', they are not 'dynamically equivalent', in the sense that they have to be revised differently. We argue that this non-equivalence should be made formal by investigating revision for logics where these two belief bases fail to be equivalent. We show that the strict implication operator '$>$', defined in modal logic by $\varphi > \psi = \Box(\varphi \to \psi)$, provides an interesting solution: the bases $\{p, q\}$ and $\{p, p > q\}$ fail to be equivalent, and the revision of $\{p, q\}$ by $\neg q$ entails $\{p, \neg q\}$ while that of $\{p, p > q\}$ by $\neg q$ entails $\{\neg p, \neg q\}$.

The paper is organized as follows. We start by recalling distance-based revision operations and modal logics (Section 2). We then formulate three simple postulates for modal revision (Section 3). After that, we introduce a general way of lifting the Hamming distance between valuations to distances between pointed S5 models (Section 4). We then propose axioms for these distances and check the status of the modal revision postulates w.r.t. the induced revision operations (Section 5). Finally, we show that if we replace material implication by strict implication then the main counterexamples against syntax-insensitive revision can be handled satisfactorily (Section 6) and conclude (Section 8).

2 Background

We suppose given a finite set $\mathsf{Var} = \{p, q, r, \dots\}$ of propositional variables. A valuation is a subset of Var. The set of all valuations is $\mathsf{Val} = \mathsf{Pow}(\mathsf{Var})$. The set PFor is the set of all propositional formulas built from Var and the connectives $\top$,

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$\neg$, and $\wedge$. The set of all **propositional models** of a formula $\varphi \in \mathsf{PFor}$ is noted $||\varphi||_{\mathsf{PC}}$.

Revision

In the original AGM framework, a revision operation maps a set of formulas B and a formula μ to a new set of formulas $B * \mu$, where the first argument of $*$ is a belief set: a set of formulas that is closed under logical theorems and modus ponens. Such sets are typically infinite. However, it has been argued in the philosophical as well as in the AI literature that this is not a realistic hypothesis, both in the case of human agents and in the case of computers: their memory can only contain finite sets of formulas that fail to be closed under theorems and modus ponens. Such finite sets are called belief bases. We follow the KM approach (Katsuno and Mendelzon 1991) and consider *formulas* instead of belief bases. In that presentation, a **revision operation** is a function $*$ mapping couples of formulas to a formula.

Let φ, ψ, μ be formulas in some logic language. Let $\vdash$ be a deduction relation for that language. A formula φ is consistent if $\varphi \nvdash \bot$; and $\varphi \equiv \psi$ abbreviates that both $\varphi \vdash \psi$ and $\psi \vdash \varphi$. The KM postulates are:

(R1) $\varphi * \mu \vdash \mu$.

(R2) If $\varphi \wedge \mu$ is consistent then $\varphi * \mu \equiv \varphi \wedge \mu$.

(R3) If μ is consistent then $\varphi * \mu$ is consistent.

(R4) If $\varphi \equiv \varphi'$ and $\mu \equiv \mu'$ then $\varphi * \mu \equiv \varphi' * \mu'$.

(R5) $(\varphi * \mu) \wedge \psi \vdash \varphi * (\mu \wedge \psi)$.

(R6) If $(\varphi * \mu) \wedge \psi$ is consistent then $\varphi * (\mu \wedge \psi) \vdash (\varphi * \mu) \wedge \psi$.

A **model-based propositional revision operation** is a function $*$ from $\mathsf{PFor} \times \mathsf{PFor}$ to $\mathsf{Pow}(\mathsf{Val})$. An important family of such operations was introduced in (Lehmann, Magidor, and Schlechta 2001; Ben-Naim 2006). All of them are built from arbitrary distances and are syntax-insensitive. The most prominent one is due to Dalal (1988) and is based on the **Hamming distance** between two valuations, which is the cardinality of their symmetric difference. Formally, the function $\mathsf{h} : \mathsf{Val} \times \mathsf{Val} \longrightarrow \mathbb{N}_0$ is such that for $u, v \in \mathsf{Val}$, $\mathsf{h}(u, v) = |(u \setminus v) \cup (v \setminus u)|$. The lifting of h to sets is a function $\mathsf{h_s}$ on $\mathsf{Val} \times \mathsf{Pow}(\mathsf{Val})$ such that

$$\mathsf{h_s}(v, U) = \begin{cases} \min\{\mathsf{h}(v, u) : u \in U\} & \text{if } U \neq \emptyset, \\ |\mathsf{Var}| & \text{otherwise,} \end{cases}$$

for $v \in \mathsf{Val}$ and $U \subseteq \mathsf{Val}$. Then **Dalal's model-based propositional revision operation** is

$$\varphi *_{\mathsf{h}} \mu = \mathsf{ArgMin}_{v \in ||\mu||_{\mathsf{PC}}} \mathsf{h_s}(v, ||\varphi||_{\mathsf{PC}}).$$

Modal Logic

There is not just one modal logic but a whole family. Here we focus on S5, our choice being mainly motivated by the simplicity of its semantics: beyond the usual Kripke models, S5 also has simpler models in terms of sets of valuations that we are going to use here. An **(S5-)model** is a non-empty subset V of Val; the set of all such models is noted $\mathsf{Mod} = \mathsf{Pow}(\mathsf{Val})$. A **pointed (S5-)model** is a pair $M = \langle V, v \rangle$ such that $V \in \mathsf{Mod}$ and $v \in V$; the set of all such models is noted PMod. The actual world v describes the current state of the world, while the elements of V describe the possible ways the world could be. For the sake of readability, in examples we underline the actual world and drop braces and commas in the description of valuations; we e.g. write $\{\underline{\emptyset}, pq\}$ instead of $\langle\{\emptyset, \{p, q\}\}, \emptyset\rangle$.

The set MFor of **modal formulas** is defined by the grammar

$$\varphi ::= p \mid \top \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\varphi \mid \Diamond\varphi$$

where p ranges over Var. The formula $\Box\varphi$ is read "φ is necessary" and $\Diamond\varphi$ is read "φ is possible". Other boolean connectives such as $\bot$, material implication $\rightarrow$, and material equivalence $\leftrightarrow$ are defined as abbreviations. A formula $\varphi \in \mathsf{MFor}$ is **propositional** when $\Box$ and $\Diamond$ do not appear in φ, that is, when $\varphi \in \mathsf{PFor}$.

The modal operator of necessity $\Box$ allows us to distinguish necessary and contingent truth of a formula φ. Necessary truths can be viewed as laws of nature, or as integrity constraints in databases. For example, if we read p_1 as "Bob is a member of Department D1" and p_2 as "Bob is a member of Department D2" then $p_1 \wedge \neg p_2$ is a contingent truth: $p_1 \wedge \neg p_2$ is presently true but is not necessarily so and may change in the future. In contrast, $\Box\neg(p_1 \wedge p_2)$ is a necessary truth expressing the integrity constraint that Bob cannot be a member of both D1 and D2.

A formula of MFor is interpreted in a pointed model $M = \langle V, v \rangle$ according to the following truth conditions:

$$\begin{array}{lll} M \Vdash p & \text{iff} & p \in v; \\ M \Vdash \Box\varphi & \text{iff} & \text{for every } v' \in V, \langle V, v' \rangle \Vdash \varphi; \\ M \Vdash \Diamond\varphi & \text{iff} & \text{for some } v' \in V, \langle V, v' \rangle \Vdash \varphi; \end{array}$$

and as usual for the boolean operators. A **pointed model of** φ is a couple $\langle V, v \rangle$ such that $\langle V, v \rangle \Vdash \varphi$; the set of all pointed models of φ is noted $||\varphi||$. If φ is without modal operators (that is, $\varphi \in \mathsf{PFor}$) then $||\varphi||$ equals the set of pointed models $\langle V, v \rangle \in \mathsf{PMod}$ such that $v \in ||\varphi||_{\mathsf{PC}}$. When $||\varphi|| \neq \emptyset$ then φ is **(S5-)satisfiable**; when $||\varphi|| = \mathsf{PMod}$ then φ is **(S5-)valid**. We denote by $\models$ the relation on MFor such that $\varphi \models \psi$ iff $||\varphi|| \subseteq ||\psi||$. Finally, a **model of** φ is a set of valuations $V \in \mathsf{Mod}$ such that $\langle V, v \rangle \Vdash \varphi$ for some $v \in V$.

Several axiomatic systems for S5 exist; we here recall one from (Chellas 1980) which extends that of propositional logic by the following axiom schemas and inference rule:

$$\Box(\varphi \wedge \psi) \leftrightarrow (\Box\varphi \wedge \Box\psi); \qquad \Box\varphi \leftrightarrow \neg\Diamond\neg\varphi;$$

$$\Box\top; \qquad \Box\varphi \rightarrow \varphi;$$

$$\frac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}; \qquad \Diamond\varphi \rightarrow \Box\Diamond\varphi.$$

Provability of a formula φ from the axioms is noted $\vdash \varphi$. The **(S5-)deduction relation** $\varphi \vdash \psi$ is defined as $\vdash \varphi \rightarrow \psi$. The axiomatics is sound and strongly complete: the relations $\vdash$ and $\models$ are identical (Chellas 1980). By $\varphi \equiv \psi$ we mean both $\varphi \vdash \psi$ and $\psi \vdash \varphi$. We say that φ is **(S5-)consistent** iff $\varphi \nvdash \bot$.

Fact 1. *Let $\varphi, \psi \in \mathsf{MFor}$. The following hold:*

1. *$\varphi \wedge \psi$ is consistent iff φ and ψ have at least one pointed model in common.*
2. *$\varphi \wedge \Diamond\psi$ is consistent iff φ and ψ have at least one model in common.*

3. *$\varphi \wedge \Diamond\psi$ is consistent iff $\Diamond\varphi \wedge \Diamond\psi$ is consistent.*
4. *If $\varphi, \psi \in \mathsf{PFor}$ then $\varphi \wedge \Diamond\psi$ is consistent iff φ is consistent and ψ is consistent.*

3 The Modal Postulates

A **modal revision operation** $*$ maps a base $\varphi \in \mathsf{MFor}$ and an input $\mu \in \mathsf{MFor}$ to a revised base $\varphi * \mu \in \mathsf{MFor}$. The new base $\varphi * \mu$ should only differ minimally from φ, and a central idea in that perspective, embodied in the KM postulate (R2), is that when φ and μ are consistent then $\varphi * \mu$ should be equivalent to $\varphi \wedge \mu$.

However, modal logic provides two perspectives on consistency: **propositional consistency** and **modal consistency**. Propositional consistency refers to cases where φ and μ share a *pointed model*. Modal consistency, on the other hand, occurs when φ and μ share a *model*, but not necessarily a pointed model. This requirement is weaker than propositional consistency. From now, when we use 'consistency' to refer to propositional consistency.

The KM postulate (R2) accounts for cases of propositional consistency but not for cases of modal consistency. We introduce three new postulates that, in conjunction with (R1)–(R6), appropriately account for revision in the context of modal consistency. The new postulates can be viewed as refinements of the preservation postulate (R2). For a start, we state a consequence of (R2).

Fact 2. *Let $*$ be a modal revision operation satisfying the KM postulates. If φ and μ are propositionally consistent and $\mu \vdash \Diamond\mu$ then $\varphi * \Diamond\mu \vdash \varphi$.*[1]

Modal consistency is more liberal than propositional consistency. For example, p and $\neg p$ are modally consistent. By Fact 1 (item 2) this is the same as consistency of $p \wedge \Diamond\neg p$. While the idea of minimal change in the case of propositional consistency involves preserving the shared pointed models, the same should hold in the case of modal consistency; only the actual world should be (minimally) changed in order to obtain a model of the input.

Let us illustrate the above point by a scenario where Detective Hercule Poirot is investigating a murder case. Initially, Poirot believes Antonio Foscarelli is innocent, because certain evidences prove that Foscarelli had no opportunity to be at the scene of the crime. But, as the investigation unfolds, those evidences proved to be false. Consequently, Poirot has to revise his beliefs accordingly, i.e., Poirot has to replace "Foscarelli is innocent" by "Foscarelli is possibly innocent". This kind of revision (i.e., replace an actual fact by a possibility) cannot be done in the classical AGM framework, which motivates our modal framework.

Our first new postulate conveys the following intuition: assume $\varphi \wedge \Diamond\mu$ is consistent; then, $\Diamond\varphi$ should be preserved when φ is revised by μ; indeed, φ entails $\Diamond\varphi$ and μ is consistent with $\Diamond\varphi$ (thanks to our assumption). More formally:

(M1) If $\varphi \wedge \Diamond\mu$ is consistent, then $\varphi * \mu \vdash \Diamond\varphi$.

When base and input are modally inconsistent, we say we have a case of **strong revision**. In contrast, when base and input are propositionally inconsistent, but modally consistent, we have a case of **weak revision**. The only things we can preserve in case of strong revision are possibilities. For example, the base $\Box p$ and the input $\neg p$ are modally inconsistent. However, $\Box p \vdash \Diamond p$ in S5, and the possibility $\Diamond p$ is propositionally consistent with $\neg p$; we therefore expect that $\Box p * \neg p \vdash \Diamond p$.

Next, our second new postulate says the following: assume φ entails ψ, and $\Diamond\mu$ is consistent with ψ; then ψ should be preserved when φ is revised by $\Diamond\mu$. More formally:

(M2) If μ propositional, $\psi \wedge \Diamond\mu$ is consistent, and $\varphi \vdash \psi$ then $\varphi * \Diamond\mu \vdash \psi$.

Note that in (M2), the restriction to propositional μ cannot be dropped at least if our modal logic is S5. Consider e.g. $\varphi = p_1 \wedge p_2$, $\psi_1 = p_1$, $\psi_2 = p_2$, and $\mu = \Box\neg(p_1 \wedge p_2)$. Then $\varphi * \Diamond\mu$ would entail both p_1 and p_2; but as $\Diamond\mu$ is S5-equivalent to $\Box(\neg p_1 \vee \neg p_2)$, this would conflict with the success postulate (R1).

We turn to our last postulate. It means the following: assume φ entails $\Diamond\psi$, and $\Diamond\mu$ is consistent with ψ; then, μ is consistent with $\Diamond\psi$; thus, $\Diamond\psi$ should be preserved when φ is revised by μ. More formally:

(M3) If μ propositional, $\psi \wedge \Diamond\mu$ is consistent, and $\varphi \vdash \Diamond\psi$, then $\varphi * \mu \vdash \Diamond\psi$.

The restriction in (M3) to propositional μ cannot be dropped, as exemplified by $\varphi = \Box(p_1 \wedge p_2)$, $\psi_1 = \Box p_1$, $\psi_2 = \Box\neg p_2$, and $\mu = \Box\neg(p_1 \wedge p_2)$.

We observe that in cases of weak revision, that is, when $\Diamond\mu$ is consistent with φ, (M2) follows from the preservation postulate (R2), and (M3) follows from (M1).

4 Distance Between Pointed S5 Models

Lifting the Hamming Distance to Sets

Contrarily to distances, pseudo-distances need not satisfy the triangle inequality: we say that a **pseudo-distance** is a function D from $\mathsf{Mod} \times \mathsf{Mod}$ to $\mathbb{N}_0$ satisfying the following **two basic axioms**:

(A1) $D(U, V) = 0$ iff $U = V$.

(A2) $D(U, V) = D(V, U)$.

The elements evaluated by D are not mere points, but subsets of Val. This allows us to formulate the following **three supplementary axioms** for D:

(A3) $D(\{u\}, \{v\}) < D(\{u'\}, \{v'\})$ iff $\mathsf{h}(u, v) < \mathsf{h}(u', v')$.

(A4) $\forall\, w \in U \setminus V$, $D(U, V \cup \{w\}) \le D(U, V) - \mathsf{h_s}(w, V)$.

(A5) $\forall\, w \in \mathsf{Val}$, $D(U, U \cup \{w\}) = \mathsf{h_s}(w, U)$.

A stronger version of (A3) was introduced in (Eiter and Mannila 1997): in the case of singletons the pseudo-distance D should equal h. (A4) says that the increased distance associated with the loss of common elements is bounded by the Hamming distance. (A5) says that if we add a new valuation to a model then the resulting model's distance from the original set is exactly the Hamming distance between this new valuation and the closest valuation from the original set. Note that the case $w \in U$ is covered by (A1).

[1] Obviously, the hypothesis that $\mu \vdash \Diamond\mu$ can be dropped when $\vdash$ is deduction in S5 because $\mu \to \Diamond\mu$ is an S5 theorem.

We now recall three specific such distances. First, the **Hausdorff distance** is the minimum of the maximum distances between corresponding points in V and U.

$$\mathsf{Dhaus}(U,V) = \max\left\{\max_{u\in U}\min_{v\in V}\mathsf{h}(u,v), \max_{v\in V}\min_{u\in U}\mathsf{h}(u,v)\right\}.$$

In (Song, Cai, and Immink 2020) a distance function based on injection is introduced. Let $\mathsf{Inj}(U,V)$ be the set of all injective functions from U to V. The **injection-based pseudo-distance** is defined by

$$\mathsf{Dinj}(U,V) = \min_{f\in\mathsf{Inj}(X,Y)}\left(\sum_{u\in X}\mathsf{h}(u,f(u)) + |\mathsf{Var}|\times(|Y|-|X|)\right),$$

where:

- $X = U$ and $Y = V$, if $|U| \leq |V|$;
- $X = V$ and $Y = U$, if $|V| < |U|$.

Finally, in (Niiniluoto 1987) the **sum-based pseudo-distance** is defined by

$$\mathsf{Dsum}(U,V) = \sum_{u\in U}\mathsf{h_s}(u,V) + \sum_{v\in V}\mathsf{h_s}(v,U).$$

Proposition 3. Dinj satisfies (A3), Dhaus satisfies (A3) and (A5), Dsum satisfies (A3)-(A5). In addition, Dhaus violates (A4), and Dinj violates (A4) and (A5).

The following table summarizes the above proposition:

	Dhaus	Dinj	Dsum
(A3)	✓	✓	✓
(A4)			✓
(A5)	✓		✓

Table 1: Satisfaction of our new pseudo-distance axioms

Lexicographic Distance between Pointed Models

We are going to view the set of accessible worlds V of a pointed model (V,v) as a first circle of possible 'fallback worlds' in a system of spheres (Lindström and Rabinowicz 1999): when a new piece of information is false at the actual world v then the 'first try' is to replace v by one of the other worlds in V. It is only when none of the elements of V satisfies the new piece of information that we have to modify V. In both cases, we are going to determine the resulting model through a distance measure.

We define the distance between two pointed models (V,v) and (U,u) to be a pair of integers:

Definition 4. Let D be a pseudo-distance. We denote by δ_D the function from PMod×PMod to $\mathbb{N}_0\times\mathbb{N}_0$. such that, $\forall M = \langle U,u\rangle, N = \langle V,v\rangle \in$ PMod, the following holds:

$$\delta_D(M,N) = \langle D(U,V), \mathsf{h}(u,v)\rangle.$$

We denote by $\leq_{\text{lex}}$ the **linear order** on $\mathbb{N}_0\times\mathbb{N}_0$ such that, $\forall\, \mathbf{r} = \langle r_1,r_2\rangle, \mathbf{s} = \langle s_1,s_2\rangle \in \mathbb{N}_0\times\mathbb{N}_0$, the following holds:

$$\mathbf{r} \leq_{\text{lex}} \mathbf{s} \text{ iff } r_1 \leq s_1 \text{ or } (r_1 = s_1 \text{ and } r_2 \leq s_2).$$

Fact 5. Let D be a pseudo-distance and $M = \langle U,u\rangle, N = \langle V,v\rangle \in$ PMod. Then:

- $u = v$ iff $\exists\, r \in \mathbb{N}_0$, $\delta_D(M,N) = \langle r,0\rangle$;
- $U = V$ iff $\exists\, r \in \mathbb{N}_0$, $\delta_D(M,N) = \langle 0,r\rangle$.

Definition 6. Let D be a pseudo-distance. We denote by Δ_D the function on Pow(PMod)×PMod such that, $\forall \mathbf{M} \subseteq$ PMod, $\forall\, N \in$ PMod, the following holds:

$$\Delta_D(\mathbf{M},N) = \min_{\leq_{\text{lex}}}\{\delta_D(M,N) : M \in \mathbf{M}\}.$$

Slightly abusing notation, we also note Δ_D the minimum distance between two sets of pointed models w.r.t $<_{\text{lex}}$.

We now explain how the lexicographical order works using an example. To do so, we will focus on Dsum since, as will be seen in the next section, it gives rise to a revision operation that satisfies all the modal postulates.

Example 7. *We have:*

$$\Delta_{\mathsf{Dsum}}(||\neg p||, ||p\wedge\Box(p\to q)||) = \langle 0,1\rangle \quad (1)$$
$$\Delta_{\mathsf{Dsum}}(||\Box p||, ||\Diamond\neg p||) = \langle 1,0\rangle \quad (2)$$
$$\Delta_{\mathsf{Dsum}}(||\Box p||, ||\neg p||) = \langle 1,1\rangle \quad (3)$$
$$\Delta_{\mathsf{Dsum}}(||\Box p||, ||\Box\neg p||) = \langle 2,1\rangle \quad (4)$$

To simplify explanations we assume that Var *equals the atoms in the formula. For item (1), the distance cannot be* $\langle 0,0\rangle$ *because the intersection of* $||\neg p||$ *and* $||p\wedge\Box(p\to q)||$ *is empty; since they are modal consistent they have at least one common model. Thus, the distance is at least* $\langle 0,n\rangle$. *For the second element of the couple,* $\{\underline{p}, pq\} \in ||\neg q||$ *and* $\{p, \underline{pq}\} \in ||p\wedge q||$, *thus we have* $\delta_{\mathsf{Dsum}}(\{\underline{p},pq\},\{p,\underline{pq}\}) = \langle 0,1\rangle$.

For item (2), $\Box p$ *has a single model,* $\underline{\{\{p\}\}}$*; and* $\Diamond\neg p$ *has two models:* $\{\emptyset\}$ *and* $\{\emptyset,\{p\}\}$. *We have* $\mathsf{Dsum}(\{\{p\}\},\{\emptyset\}) = 2$ *while* $\mathsf{Dsum}(\{\{p\}\},\{\emptyset,\{p\}\}) = 1$. *The actual world remains the same, therefore* $\Delta_{\mathsf{Dsum}}(||\Box p||, ||\Diamond\neg p||) = \langle 1,0\rangle$.

The explanation of item (3) does not differ from that of item (2) as far as the first component of Δ_{Dsum} *is concerned. As to the second component,* $\neg p$ *being logically stronger than* $\Diamond\neg p$ *we have to also shift the actual world from* $\{p\}$ *to* $\emptyset$. *Hence,* $\Delta_{\mathsf{Dsum}}(||\Box p||, ||\neg p||) = \langle 1,1\rangle$. *In Figure 1 a graphical representation of this item is given.*

For item (4), $\Box p$ *and* $\Box\neg p$ *have exactly one model, specifically* $\{\{p\}\}$ *for the former and* $\{\emptyset\}$ *for the latter. Thus,* $\Delta_{\mathsf{Dsum}}(||\Box p||, ||\Box\neg p||) = \langle 2,1\rangle$.

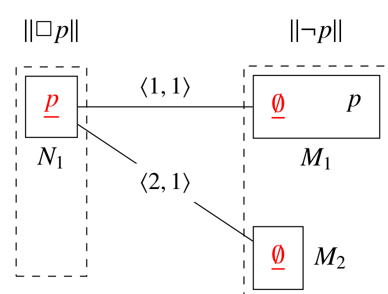

Figure 1: Illustration of Example 7, item (3); each box represents a pointed model, with its actual world underlined.

5 Distance-Based Revision Operation for S5

The following definition generalizes Dalal's revision operation from propositional logic to modal logic.

Definition 8. A **model-based modal revision operation** is a function $*$ from $\mathsf{MFor} \times \mathsf{MFor}$ to $\mathsf{Pow}(\mathsf{PMod})$. Let D be a pseudo-distance. We denote by $*_D$ the model-based modal revision operation such that, $\forall\, \varphi, \mu \in \mathsf{MFor}$:

$$\varphi *_D \mu = \{M \in \|\mu\| \ : \forall\, N \in \|\mu\|, \\ \Delta_D(\|\varphi\|, M) \leq_{\mathsf{lex}} \Delta_D(\|\varphi\|, N)\}.$$

Let us exemplify the above definition with $*_{\mathsf{Dsum}}$ as $*_D$.

Example 9. *Consider the following revision results (for each of them,* Var *consists of the atoms appearing in the base and/or input):*

$$(p \wedge q) *_{\mathsf{Dsum}} \neg q = \|p \wedge \neg q \wedge \Diamond(p \wedge q)\|$$
$$(p \wedge q) *_{\mathsf{Dsum}} \Diamond\neg q = \|p \wedge q \wedge \Diamond\neg q\|$$
$$\Box q *_{\mathsf{Dsum}} \neg q = \|\neg q \wedge \Diamond q\|$$
$$(\Box q \wedge p) *_{\mathsf{Dsum}} \neg q = \|p \wedge \neg q \wedge \Box(p \vee q) \wedge \Diamond(p \wedge q)\|$$

The outcome of the revision depends on the set Var. For example, $\Box\neg p *_{\mathsf{Dsum}} \Box p$ is $\|\Box p\|$ for $\mathsf{Var}_1 = \{p\}$, while it equals $\|\Box p \wedge (\Box q \vee \Box\neg q)\|$ for $\mathsf{Var}_2 = \{p, q\}$. Related to that, $*_{\mathsf{Dsum}}$ fails to satisfy Parikh's principle of language splitting (Kourousias and Makinson 2007; Parikh 1999), which says that if the variables of φ and ψ are disjoint and the variables of μ are a subset of those of φ then $(\varphi \wedge \psi) * \mu = (\varphi * \mu) \wedge \psi$. This is illustrated by $\varphi = \Box\neg p$, $\psi = \Diamond q \wedge \Diamond\neg q$, and $\mu = \Box p$: we have $(\varphi \wedge \psi) *_{\mathsf{Dsum}} \mu = \|\Box p\|$, $(\varphi *_{\mathsf{Dsum}} \mu) \cap \|\psi\| = \|\bot\|$.

Figure 2 illustrates the distinction between weak and strong revision that we have introduced in Section 3. For the former, the base and the input share a model; hence the distance between them is $\langle 0, n\rangle$ for some $n \geq 1$. Then minimal change means to keep the model and modify the actual world. In the case of strong revision the base and the input do not share any model; hence the minimum distance is $\langle n, m\rangle$ for some $n \geq 1$ and $m \geq 0$.

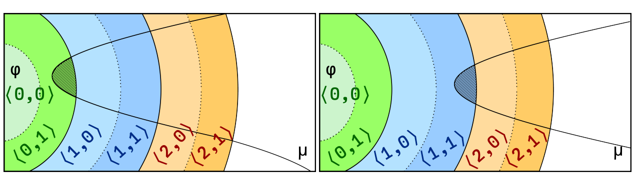

Figure 2: Weak revision (left) and strong revision (right); each colored band represents the distance between the pointed models of φ and those μ; the intersection shows the revision result.

Satisfaction of the AGM Postulates

To show that $*_D$ satisfies the AGM postulates we can use the results proved in (Katsuno and Mendelzon 1991) that use the notion of **faithful assignment**. Given the set PMod, consider a function that assigns to each formula $\varphi \in \mathsf{MFor}$ a total order $\leq_\varphi$ over PMod. This assignment is said to be faithful if for any $M, M' \in \mathsf{PMod}$ the following hold:

1. If $M, M' \in \|\varphi\|$ then $M <_\varphi M'$ does not hold.
2. If $M \in \|\varphi\|$ and $M \notin \|\varphi\|$ then $M <_\varphi M'$.
3. If $\varphi \equiv \psi$ then $\leq_\varphi = \leq_\psi$.

Let $\mathbf{M} \subseteq \mathsf{PMod}$ and $M \in \mathsf{PMod}$. Then $M \in \mathbf{M}$ is minimal in $\mathbf{M}$ under the ordering $\leq_\varphi$ if and only if there exists no other $M' \in \mathbf{M}$ such that M is strictly less than M' according to $\leq_\varphi$. The set of minimal elements of $\mathbf{M}$ w.r.t. $\leq_\varphi$ is $\min_{\leq_\varphi}(\mathbf{M})$.

Let D be a pseudo-distance. We now define a faithful assignment by associating to every φ a total order $\leq_\varphi$ such that $M \leq_\varphi M'$ if and only if $\Delta_D(\|\varphi\|, M) \leq_{\mathsf{lex}} \Delta_D(\|\varphi\|, M')$. Thus, Definition 8 can as well be formulated as: $\varphi *_D \mu = \min_{\leq_\varphi}(\|\mu\|)$. Since the assignment is faithful, the satisfaction of the AGM postulates can be derived from the results proved in (Katsuno and Mendelzon 1991).

Satisfaction of the Modal Postulates

Our main result is a link between the distance axioms and the revision postulates. Loosely speaking, this link is the following: if a set pseudo-distance satisfies certain axioms, then the induced revision operation satisfies (M1-M3).

To formally state our main result, we need some technical preliminaries.

Lemma 10. Let D be a pseudo-distance satisfying (A1). Let $\varphi, \mu \in \mathsf{MFor}$ such that $\varphi \wedge \Diamond\mu$ is consistent. Then:

$$\varphi *_D \mu = \{\langle U, u\rangle \in \|\mu\| : \exists\, \langle V, v\rangle \in \|\varphi\|, U = V \text{ and} \\ \mathsf{h}(u, v) = \min(X)\},$$

where:

$$X = \{r \in \mathbb{N} \ : \exists\, \langle U', u'\rangle \in \|\mu\|, \exists\, \langle V', v'\rangle \in \|\varphi\|, \\ V' = U' \text{ and } r = \mathsf{h}(u', v')\}.$$

Definition 11. We denote by h_f the function from $\mathsf{Val} \times \mathsf{MFor}$ to $\mathbb{N}_0$ such that, $\forall\, v \in \mathsf{Val}$, $\forall\, \varphi \in \mathsf{MFor}$,

$$\mathsf{h}_\mathsf{f}(v, \varphi) = \min\{r : \exists\, \langle U, u\rangle \in \|\varphi\|, \exists\, u' \in U, r = \mathsf{h}(v, u')\}.$$

Lemma 12. Let D be a pseudo-distance satisfying (A4) and (A5). Let $\varphi, \mu \in \mathsf{MFor}$ such that μ is propositional and $\varphi \wedge \Diamond\mu$ is inconsistent. Then:

$$\varphi *_D \Diamond\mu = \{N \in \|\mu\| : \exists\, M = \langle U, u\rangle \in \|\mu\|, \\ \exists\, O = \langle V, v\rangle \in \|\varphi\|, \mathsf{h}_\mathsf{s}(u, V) = \mathsf{h}_\mathsf{f}(u, \varphi) \\ \text{and } N = \langle V \cup \{u\}, v\rangle\}.$$

Lemma 13. Let D be a pseudo-distance satisfying (A1), (A4) and (A5). Let $\varphi, \mu \in \mathsf{MFor}$ such that μ is propositional and $\varphi \wedge \Diamond\mu$ is inconsistent. Then:

$$\varphi *_D \mu = \{N \in \|\mu\| : \exists\, M = \langle U, u\rangle \in \|\mu\|, \\ \exists\, O = \langle V, v\rangle \in \|\varphi\|, \mathsf{h}(v, u) = \mathsf{h}_\mathsf{f}(u, \varphi) \\ \text{and } N = \langle V \cup \{u\}, u\rangle\}.$$

We are ready to state the main result of the paper:

Proposition 14. Let D be a pseudo-distance.

- If D satisfies (A1) then $*_D$ satisfies (M1).
- If D satisfies (A4) and (A5) then $*_D$ satisfies (M2).
- If D satisfies (A1), (A4), and (A5) then $*_D$ satisfies (M3).

Proof. We only give the proof of the first item. Let D be any pseudo-distance between sets of valuations satisfying (A1) and let $*_D$ be the resulting operation. Suppose that $\Diamond\mu$ is consistent with φ. It follows from this that there exists a pair (V, v) and another (U, u) such that $(V, v) \Vdash \varphi$, $(U, u) \Vdash \mu$ and $V = U$. As D satisfies (A1), $D(V, U) = 0$; in addition, $\Delta_{\mathsf{Dsum}}(\|\varphi\|, \|\mu\|) = \langle 0, n\rangle$ for some n. But indeed, since V is a S5-model and $(V, v) \Vdash \varphi$, we have that for any $u' \in U$, $(U, u') \Vdash \Diamond\varphi$. Thus $\varphi *_D \mu \Vdash \Diamond\varphi$. □

Proposition 15. Both $*_{\mathsf{Dhaus}}$ and $*_{\mathsf{Dinj}}$ satisfy (M1). $*_{\mathsf{Dsum}}$ satisfies (M1)-(M3). In addition, both $*_{\mathsf{Dhaus}}$ and $*_{\mathsf{Dinj}}$ violate (M2) and (M3).

The following table summarizes the above proposition:

	$*_{\mathsf{Dhaus}}$	$*_{\mathsf{Dinj}}$	$*_{\mathsf{Dsum}}$
M1	✓	✓	✓
M2			✓
M3			✓

Table 2: Satisfaction of our new modal postulates.

6 Syntax-Sensitive Revision and Strict Implication

One of the most appealing arguments for the use of belief bases is that while $\{p, q\}$ and $\{p, p \to q\}$ are logically equivalent in classical propositional logic, they should however not be revised in the same way: the revision of the former by $\neg q$ should result in $\{p, \neg q\}$ because p and q are independent.

In contrast, the revision of $\{p, p \to q\}$ by $\neg q$ should lead to $\{p \to q, \neg q\}$. The latter is logically incomparable with $\{p, \neg q\}$, in the sense that neither implies the other. A variant of the argument can be formulated in terms of material equivalence and can be found in the SEP entry for belief revision (Hansson 2022). This variant is based on the fact that the bases are *statically equivalent* but not *dynamically equivalent*.

The distinction goes beyond belief revision and relates to the debate between coherentists and foundationalists: the former view belief states as having a 'flat' structure where each belief has the same status, while the latter view beliefs as being organized in a web where some beliefs are more basic and support other, less basic beliefs.

A Plea for a Logical Analysis

Syntax-sensitive revision operations do not satisfy (R4). They exploit the 'dynamical non-equivalence' of $\{p, q\}$ and $\{p, p \to q\}$ as well as that of $\{p, q\}$ and $\{p, p \leftrightarrow q\}$, but do not explain it. Indeed, a natural question to ask is whether one can give a logical foundation to that distinction.

Intuitively, the material implication $p \to q$ is too weak to adequately capture the 'if-then' link between p and q; and similarly for the material equivalence $p \leftrightarrow q$. Actually, Bob's belief that $p \leftrightarrow q$ (that Ms. Smith will conform to the party decision) has a dispositional nature: Bob's belief that $p \leftrightarrow q$ is not only about here and now, that is when it is the case that the party supports the proposal, but also about the situation where the party does not support the proposal.

This leads to the study of logics with a stronger implication connective '→', such that $p \wedge q$ and $p \wedge (p \to q)$ fail to be equivalent: can such logics be equipped with revision operations that satisfy syntax independence? The first candidate is intuitionistic logic, whose implication is stronger than the material implication that of classical logic.

However, our two belief bases are logically equivalent there, too. Relevance logics are more interesting: the formulas $p \wedge q$ and $p \wedge (p \to q)$ typically fail to be equivalent there. Relevance logics however have a paraconsistent nature: basically any set of formulas is consistent, and a first difficulty is to design a revision operation that avoids collapsing into expansion.

Several proposals had been made in the literature (Restall and Slaney 1995; Lakemeyer and Lang 1996); the recent (Schwind, Konieczny, and Pino Pérez 2022) considers Priest's Logic of Paradox LP.

We here take another research avenue that, as far as we know, has not been tried yet and that we believe to be promising: the logic of strict implication.

Revision with Strict Implication

An operator of strict implication can be defined in modal logic as $\varphi > \psi = \Box(\varphi \to \psi)$ (Lewis and Langford 1959; Hughes and Cresswell 1968). Therefore its truth condition is:

$$(V,v) \Vdash \varphi > \psi \text{ if for every } v' \in V\text{: } (V,v') \Vdash \varphi \text{ implies } (V,v') \Vdash \psi.$$

The modal operator of necessity can be defined from strict implication: $\Box\varphi$ abbreviates $\top > \varphi$. For example, the set of pointed models of $p \wedge (p > q)$ is

$$\|p \wedge (p > q)\| = \{(V,v) : V \subseteq \mathsf{Pow}(\mathsf{Var}), v \in V, p \in v \text{ and for every } v' \in V : p \in v' \text{ implies } q \in v\}.$$

Hence $\|p \wedge (p > q)\| = \{\{\underline{pq}, q, \emptyset\}, \{\underline{pq}, q\}, \{\underline{pq}, \emptyset\}, \{\underline{pq}\}\}$ for the case $\mathsf{Var} = \{p, q\}$.

The material implication $(p \wedge (p > q)) \to (p \wedge q)$ and its strict variant $(p \wedge (p > q)) > (p \wedge q)$ are both valid. In contrast, neither $(p \wedge q) \to (p \wedge (p > q))$ nor $(p \wedge q) > (p \wedge (p > q))$ are valid.

It is exactly this non-equivalence of $p \wedge (p > q)$ and $p \wedge q$ that makes the logic of strict implication an interesting basic logic for belief revision.

The following example illustrates the above point:

Example 16. *Assume* $\mathsf{Var} = \{p, q\}$.

$$p \wedge q *_{\mathsf{Dsum}} \neg q = \|p \wedge \neg q \wedge \neg(p > \neg q)\| \quad (5)$$

$$p \wedge (p > q) *_{\mathsf{Dsum}} \neg q = \|\neg p \wedge \neg q \wedge p > q \wedge \neg(p > \neg q)\| \quad (6)$$

$$p \wedge (p > q) *_{\mathsf{Dsum}} \neg p = \|\neg p \wedge q \wedge p > q \wedge \neg(p > \neg q)\| \quad (7)$$

$$p > q *_{\mathsf{Dsum}} q > p = \|(p > q) \wedge (q > p)\| \quad (8)$$

$$p \wedge \neg q *_{\mathsf{Dsum}} p > q = \|(p > q) \wedge (p \vee \neg q)\| \quad (9)$$

$$p > q *_{\mathsf{Dsum}} \neg(p > q) = \|p \to q \wedge (\neg(\neg q > p) \vee \neg(q > \neg p))\| \quad (10)$$

Items 5 and 6 illustrate that $*_{\mathsf{Dsum}}$ *satisfies the intuitions we have put forward in the introduction.*

In item 7, unlike the previous item, q should be true in the actual world.

Item 8 is implied by the AGM preservation postulate: as $q > p$ *is consistent with* $p > q$*, the revision of the former by the latter is equivalent to their conjunction.*

All the examples in items 5, 6, 7 and 8 are weak revisions. The next two items 9 and 10 showcase strong revision.

For Item 9, except for the pointed model $\{\underline{p}\}$*, for any other* $(U, u) \in \|p > q\|$ *there is a pointed model* $(\overline{V}, v) \in \|p \wedge \neg q\|$ *such that* $\mathsf{Dhaus}(V, U) = 1$. *This implies that the set of closest pointed models is determined by the distance between their actual worlds. Hence, the closest actual world after the revision is either* $\{p, q\}$ *or* $\emptyset$.

For Item 10, the revision process involves adding a new world that is not present in any model of $p > q$*: the world where* $p \wedge \neg q$ *is true, that is,* $\{p\}$. *While the distance between* $\{p\}$ *and* $\{q\}$ *is 2, the distance between* $\{p\}$ *and* $\{pq\}$ *is only 1; and the same is the case for the distance between* $\{p\}$ *and* $\emptyset$. *Hence, for all models of* $p > q$ *containing one of these worlds we can reduce the lexical distance to its minimum.*

7 Related Work

We have already mentioned that there are only relatively few studies of revision operations for modal logics. Probably one reason for that is Fuhrmann's finding that several AGM postulates make little sense when bases are required to satisfy the (Poss) principle (Hansson 1999; Fuhrmann 1989). This principle says that when ψ is consistent with φ then $\varphi \vdash \Diamond\psi$. As shown in (Fuhrmann 1989), it is incompatible with the AGM revision postulates (in particular the preservation postulate (R2)). Our approach is immune to that negative result because we do not suppose that belief bases satisfy (Poss).

Closest to our work is an approach by Caridroit et al. It is based on the work in (Ågotnes, van der Hoek, and Wooldridge 2012) and also rejects (Poss) (Caridroit et al. 2016). Caridroit et al. take a doxastic perspective where accessible worlds model an agent's belief state. They consider that two pointed models (V,v) and (U,u) agreeing on the actual world (in our terms: $v = u$) have smaller distance than models disagreeing on the actual world.

Our contribution takes the opposite view: agreement on the actual world is less important than agreement on the set of accessible worlds, as embodied in our lexicographic ordering on pointed models $\leq_{\mathsf{lex}}$. As our distance differs from Caridroit et al.'s, the resulting revision operation is different from theirs. As far as we know, the only study of revision operations for the logic of strict implication is (Berto 2019). It is not built on a classical propositional logic basis because it assumes hyperintensionality of beliefs. It is therefore difficult to compare it to our approach.

In a spirit similar to ours, Konieczny et al. proposed what may be called a logic for syntax-sensitive revision (Konieczny, Lang, and Marquis 2005). They introduce an extension of propositional logic where the belief bases $\{p, q\}$ and $\{p \wedge q\}$ are not logically equivalent. This is obtained by giving the status of a logical connective to the set comma and allows them to define an operation where the revision of these bases by $\neg q$ has different outcomes.

Finally, belief revision and merging operations under integrity constraints as studied in (Konieczny and Pino Pérez 2002) can also be related to our approach: we can identify integrity constraints as formulas prefixed by a modal operator of necessity. However, in the approach of (Konieczny and Pino Pérez 2002) integrity constraints cannot be revised, while our approach caters to that: the input can not only be inconsistent with the integrity constraints, but we can also revise by a new integrity constraint.

The topic of truthlikeness as studied in philosophy of science can be related to our study of distances between sets of possible worlds. In (Eiter and Mannila 1997), Eiter and Manila propose different definitions of distances based on the arguments given in (Oddie 1978; Niiniluoto and Tuomela 1979). Other distances between sets are suggested in (Fujita 2013; Song, Cai, and Immink 2020).

8 Conclusion

In this paper, we have started a systematic study of belief revision operations for modal logic. Syntactically, we have introduced novel modal postulates that ensure minimal change; semantically, we have proposed a family of lexicographical distances between pointed models that extend the traditional Hamming distance. We have also proposed a specific instance of this distance, yielding the revision operation $*_{\mathsf{Dsum}}$. We have shown that it sheds new light on some criticisms of AGM revision that had led to the development of syntax-sensitive revision operations.

Our work could be extended in at least three directions. First, we could consider logics other than S5. This requires a careful extension of our lexicographic (pseudo-)distance. It is straightforward for the modal logics KD45 and K45, while it is less obvious for logics such as K and S4. Furthermore, beyond other logics of strict implication, one may view strict implication as a particular conditional operator and exploit that in many conditional logics the formulas $p \wedge q$ and $p \wedge (p > q)$ fail to be equivalent.

Precisely, the two formulas are only logically equivalent in those of Lewis's sphere systems (Lewis 1973) satisfying both weak and strong centering, that is, the actual world is the only element of the innermost sphere. Indeed, such models validate the axiom of conditional modus ponens $(p \wedge (p > q)) \rightarrow q$ and the axiom of conjunctive sufficiency $(p \wedge q) \rightarrow (p > q)$, and from these two one can prove the logical equivalence of $p \wedge (p > q)$ and $p \wedge q$.

Second, we could move from the Hamming distance to the set inclusion-based ordering underlying Satoh's revision operation (Satoh 1988) and Winslett's update operation (Winslett 1988, 1990).

Third, it would be interesting to have a complete set of postulates characterising our operation. This is however certainly not easy to achieve: our operation extends Dalal's belief revision operation, for which as far as we know no characterisation exists in the literature.

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