Replicating Electoral Success

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Abstract

A core tension in the study of plurality elections is the clash between the classic Hotelling–Downs model, which predicts that two office-seeking candidates should cater to the median voter, and the empirical observation that democracies often have two major parties with divergent policies. Motivated in part by this tension, we introduce a dynamic model of candidate positioning based on a simple bounded rationality heuristic: candidates imitate the policy of previous winners. The resulting model is closely connected to evolutionary replicator dynamics. For uniformly-distributed voters, we prove in our model that with $k = 2$, 3, or 4 candidates per election, any symmetric candidate distribution converges over time to the center. With $k \geq 5$ candidates per election, however, we prove that the candidate distribution does not converge to the center and provide an even stronger non-convergence result in a special case with no extreme candidates. Our conclusions are qualitatively unchanged if a small fraction of candidates are not winner-copiers and are instead positioned uniformly at random in each election. Beyond our theoretical analysis, we illustrate our results in extensive simulations; for five or more candidates, we find a tendency towards the emergence of two clusters, a mechanism suggestive of Duverger's Law, the empirical finding that plurality leads to two-party systems. Our simulations also explore several variations of the model, where we find the same general pattern: convergence to the center with four or fewer candidates, but not with five or more. Finally, we discuss the relationship between our replicator dynamics model and prior work on strategic equilibria of candidate positioning games.

Extended version — https://arxiv.org/abs/2402.17109

Code — https://github.com/tomlinsonk/plurality-replicator-dynamics

Introduction

In a democracy, election outcomes determine the trajectory of public policy. A central question in the study of elections is therefore whether we can model which policies are electorally successful. However, elections are extremely complex, with layered interactions between voters, candidates, and the incentives that guide them. To understand the principles governing elections, it is valuable to pare down this

complexity and study simple models. The literature around this topic traces its roots to Hotelling (1929) and Downs (1957). In the Hotelling–Downs model, candidates compete for election in a one-dimensional policy space. Under the assumption that voters prefer candidates with closer policies, two rational office-seeking candidates will adopt the policy of the median voter, since any other position receives strictly fewer votes. Thus, the central prediction of the Hotelling–Downs model is that we should expect two competing candidates to espouse near-identical moderate policies. However, this is not what we observe in modern democracies: countries using plurality often have two dominant parties with markedly different policies (Poole and Rosenthal 1984; Grofman 2004; Riker 1982). Decades of research have attempted to reconcile this observed policy divergence with the intuitive arguments of Hotelling and Downs (Grofman 2004; Osborne 1995; Palfrey 1984; Wittman 1983).

The majority of this work has assumed that candidates are rational and able to make strategically optimal decisions. However, the growing literature on bounded rationality (Simon 1955, 1979) and decision-making heuristics (Tversky and Kahneman 1974), as well as the complexity of elections, casts doubt on whether this is likely in practice. In a notable exception to the literature, Bendor et al. (2011) argue that heuristics play a crucial role in electoral strategy:

> Campaigns are of chess like complexity—worse, probably: instead of a fixed board, campaigns are fought out on stages that can change over time, and new players can enter the game. Hence, cognitive constraints (e.g., the inability to look far down the decision tree, to anticipate your opponent's response to your response to their response to your new ad) inevitably matter. [...] Thus, political campaigns, like military ones, are filled with trial and error. A theme is tried, goes badly (or seems to), and is dropped. [...] *In short, there are good reasons for believing that the basic properties of experiential learning—becoming more likely to use something that has worked in the past and less likely to repeat something that has failed—hold in presidential campaigns.* (Bendor et al. 2011, emphasis ours)

Our model. In this paper, we introduce a model of candidate positioning based on such a heuristic: candidates
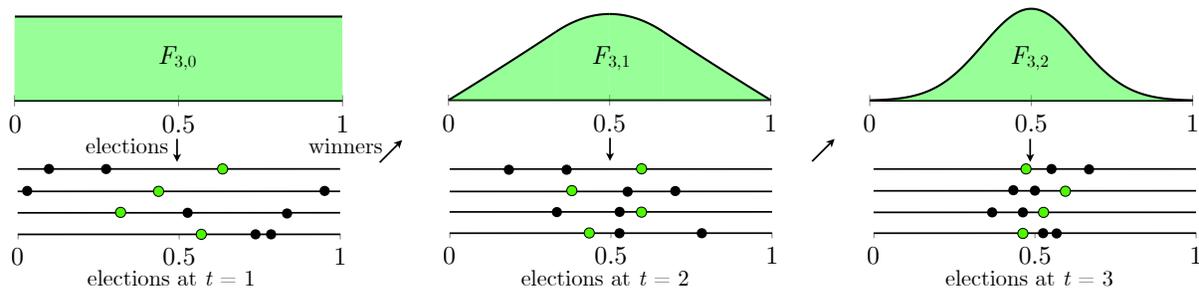
Figure 1: Replicator dynamics for candidate positioning with $k = 3$ candidates per election. The top row shows the winner distributions $F_{k,t}$ for each generation t, starting from a uniform distribution at $t = 0$, while the bottom row shows four example elections per generation. In each generation, candidates sample their positions from the winner distribution from the previous generation. Plurality winners (with voters uniform over $[0, 1]$) are indicated in green.

imitate success. We focus on plurality elections, where each voter casts one vote and the candidate with the most votes wins. We assume voters have 1-Euclidean preferences (Coombs 1950; Elkind, Lackner, and Peters 2016), where voters and candidates occupy points in the unit interval $[0, 1]$ and voters prefer closer candidates. To represent a large voting population, our model uses a continuum of voters and continuous vote shares rather than discrete counts. Diverging from prior work, we model a large number of k-candidate elections that proceed in generations. In each generation, candidates copy the policy position of a winner from the previous generation, a heuristic that political scientists have argued is prevalent in real-world elections (Shipan and Volden 2008; Böhmelt et al. 2016; Ezrow et al. 2021). As with voters, our model uses a continuous distribution of candidate positions in each generation, which can be viewed as either capturing the expected behavior of a finite number of elections or as the infinite-election limit. See Figure 1 for a visualization of this model. Our assumptions yield a mathematical model equivalent to the well-studied *replicator dynamics* from evolutionary biology (Taylor and Jonker 1978; Schuster and Sigmund 1983). In the classic replicator dynamics, n strategies (or alleles) compete in a population, increasing in prevalence at a rate proportional to their average fitness in pairwise contests drawn from the current population. Our model arises from taking such dynamics and moving to a continuous strategy space with k-way interactions in discrete time (i.e., k-candidate elections), treating the plurality win probability as fitness; we therefore refer to it as *replicator dynamics for candidate positioning*.

Our results. Our main results characterize the long-run behavior of the replicator dynamics for different values of k, the number of candidates per election. We find a dramatic change in the dynamics as k increases. We focus on initial candidate distributions that are symmetric and have a continuous CDF and assume voters are uniform over $[0, 1]$ (we find in simulation that the same patterns hold with other symmetric voter distributions). When $k = 2$, we prove that the candidate distribution converges to a point mass at $1/2$ under the replicator dynamics (Theorem 2), just like rational candidates in the Hotelling–Downs model. However, we also prove central convergence in our model for $k = 3$

(Theorem 3) and $k = 4$ (Theorem 4), in contrast to three- and four-candidate extensions of the Hotelling–Downs argument (Cox 1987). Surprisingly, we prove that the pattern ends there: for any $k \geq 5$, the candidate distribution does not converge to $1/2$ (Theorem 5). Our proof provides intuition about why this occurs: when candidates begin to converge to the center, only the left- and right-most can win, as the others are "squeezed out." With many candidates, these extremal winners are likely far from the center, making the next generation more polarized. See Figure 2 for simulations demonstrating the patterns from our theorems. These simulations reveal a tendency for candidate counts larger than 4 to result in two distinct clusters of policies (around $1/4$ and $3/4$ with uniform voters). This is reminiscent of *Duverger's Law* (Duverger 1959; Riker 1982), the observation that plurality elections tend towards two-party systems.

Additionally, our results are robust to noise: even if some candidates are positioned uniformly at random, we can still show (approximate) convergence to the center for $k = 2, 3, 4$ and non-convergence for $k \geq 5$ (Theorem 6). While we are not able to derive the asymptotic distribution for $k \geq 5$ in general, we show that when the initial candidate distribution is supported only on $(1/4, 3/4)$, the candidate density in an interval around $1/2$ goes to 0 (Theorem 10). We explore variants of the model in simulation, including other voter distributions, noisy position-copying, memory of prior rounds, and mixtures of candidate counts. Across these variants, we observe the same pattern as in our main result.

We conclude by relating our model back to traditional analyses of Nash equilibria. The close relationship between replicator dynamics fixed points and Nash equilibria is well-known (Hofbauer and Sigmund 2003), but we argue that ignoring dynamics and focusing only on equilibria can lead to brittle conclusions. In particular, we show that different assumptions on voter behavior when candidates occupy the same points lead to dramatically different Nash equilibria than previously reported (Cox 1987); in contrast, this has no effect on our replicator dynamics results.

Related work. Most research on candidate positioning has focused on strategic games, both one-shot (Kurella 2017; Bol, Dellis, and Oak 2016; Osborne 1995; Weber 1992; Palfrey 1984; Wittman 1983) and multi-round (Duggan and
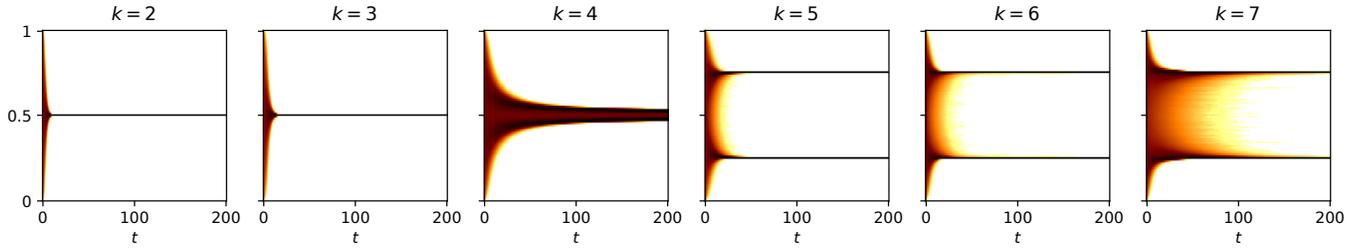
Figure 2: Replicator dynamics runs for $k = 2, \ldots, 7$ and 200 generations. Each plot shows 50 runs stacked on top of each other; each run has 100,000 elections per generation. Darker regions indicate higher candidate density with a log-scaled colormap. As our theory establishes, the candidate distribution converges to the center for $k = 2, 3, 4$, but not for $k \geq 5$.

Martinelli 2017; Nunnari and Zápal 2017; Forand 2014; Chappell and Keech 1986; Rosenthal 1982; Kramer 1977; Wittman 1977), including in the computational social choice community (Feldman, Fiat, and Obraztsova 2016; Harrenstein et al. 2021). There has been some work on boundedly-rational candidates (Kollman, Miller, and Page 1992, 1998; Bendor et al. 2011). Our paper is set apart in our replicator dynamics approach, and our success deriving analytical results for more than two candidates. We are aware of one paper (Laslier and Ozturk Goktuna 2016) combining a spatial model of elections and replicator dynamics, but the number of parties is fixed to two and the focus is instead on competition between office- and policy-motivated party members.

Replicator Dynamics for Candidate Positioning

We now formally introduce our model. We consider a one-dimensional policy space: the unit interval $[0, 1]$. Candidates and voters reside at points in the interval. To model a large voting population and for tractability, we assume voters are uniform over $[0, 1]$, but we later relax this assumption in simulation. We assume voters have 1-Euclidean preferences (Elkind, Lackner, and Peters 2016)—that is, they vote for the closest candidate. The *vote share* of a candidate i is the fraction of voters who vote for i. With uniform voters, the vote share of a candidate is equal to half the distance between the candidates to its left and right (a candidate adjacent to a boundary gets the entire vote share on its boundary side). Under plurality voting, the candidate with the largest vote share wins; in the case of tied maximum vote shares, the tie is broken uniformly at random.

Our replicator dynamics model of candidate positioning supposes that elections proceed in generations $t = 1, 2, \ldots$, with (infinitely) many elections per generation. We assume the number of candidates in each election is fixed at k (later, we relax this assumption in simulation). The core idea of our model is that candidates in generation t chose their policy positions by copying the position of a winner from the previous generation $t - 1$. More formally, let F_0 be the initial candidate distribution and let $F_{k,t}$ denote the distribution of winner positions in generation t with k candidates per election. We define $F_{k,0} = F_0$ for all k, although we usually write F_0 since the initial distribution does not depend on k. In generation t, each election consists of k candidates

with positions $X_{1,t}, \ldots, X_{k,t} \sim F_{k,t-1}$. We use $F_{k,t}(x)$ to denote the CDF of the winner distribution in generation t and $f_{k,t}(x)$ to denote the PDF. Let $\mathrm{Plur}(X_{1,t}, \ldots, X_{k,t})$ be the position of the plurality winner given candidate positions $X_{1,t}, \ldots, X_{k,t}$ and uniformly distributed voters.

Definition 1. Given an initial candidate distribution F_0 and a candidate count k, the *replicator dynamics for candidate positioning* (under plurality with uniform 1-Euclidean voters) are, for all $t > 0$,

$$F_{k,t}(x) = \Pr(\mathrm{Plur}(X_{1,t}, \ldots, X_{k,t}) \leq x), \qquad (1)$$
$$X_{i,t} \sim F_{k,t-1}, \ \forall i = 1, \ldots, k.$$

Or, in terms of the PDF:

$$f_{k,t}(x) = k \Pr(\mathrm{Plur}(x, X_{2,t} \ldots X_{k,t}) = x) f_{k,t-1}(x). \ (2)$$

This model is closely linked to evolutionary replicator dynamics (Taylor and Jonker 1978; Schuster and Sigmund 1983), where there are n strategies which increase in frequency proportionally to their fitness against the current population. This is exactly what Equation (2) captures in our continuous setting: strategy x increases in density proportional to its plurality win rate against the current population.

The main question we study is how the candidate distribution evolves over time under the replicator dynamics. We focus on cases where F_0 is symmetric about $1/2$ and contains no point masses; we call such distributions *symmetric* and *atomless* (ensuring that the probability multiple candidates share the exact same point is 0). Since we assume F_0 is symmetric, all subsequent winner distributions are also symmetric by the symmetry of plurality with a uniform voter distribution—we make heavy use of this fact in our analysis. Some of our results require an additional assumptions on F_0. We say F_0 is *positive near 1/2* if $F_0(x) < 1/2$ for all $x < 1/2$ (equivalently, $f_0(x) > 0$ in an interval around $1/2$, by symmetry). We define $\mathcal{F}$ to be the set of all symmetric and atomless distributions over $[0, 1]$ and $\mathcal{F}^+ \subset \mathcal{F}$ to be the subset of such distributions which are also positive near $1/2$.

In this section, we prove our main result piece-by-piece.

Theorem 1. *Let $F_0 \in \mathcal{F}^+$. For $k \in \{2, 3, 4\}$, the candidate distribution converges to a point mass at $1/2$ under the replicator dynamics. In contrast, for $k \geq 5$, the candidate distribution does not converge to a point mass at $1/2$.*

Theorem 1 follows from Theorems 2 to 5. Our results for $k \in \{2, 3, 4\}$ give fine-grained characterizations of the dy-
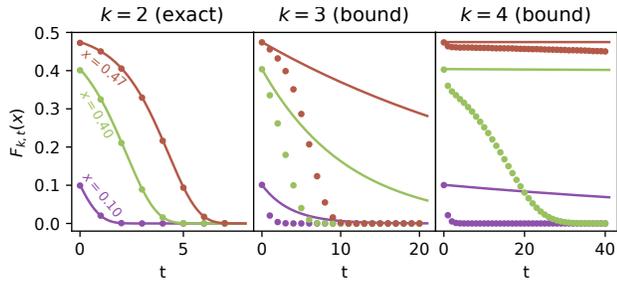
Figure 3: Simulations corroborating our convergence results in Theorems 2 to 4, showing the simulated candidate distribution CDF at various points x (dots) alongside the theoretical predictions (lines). These simulations use 50 trials with 100,000 elections per generation, no noise, and enhanced symmetry (see Simulation section).

namics, which imply convergence to the center. Our non-convergence result offers less insight into the dynamics for $k \geq 5$, but in a following section we prove a stronger result in the special case where F_0 has no extreme candidates. All proofs omitted for space are available in the extended version (Tomlinson et al. 2024).

Two-candidate plurality with symmetric voters is simple: whichever candidate is closer to $1/2$ has the larger vote share and wins. This idea translates into a closed form for $F_{2,t}$.

Theorem 2. *Let $F_0 \in \mathcal{F}$. For all $x < 1/2$ and $t \geq 0$,*
$$F_{2,t}(x) = [2 \cdot F_0(x)]^{2^t} / 2.$$

Thus, at any point x with $F_{2,0}(x) < 1/2$, $F_{2,t}(x)$ rapidly goes to 0, so the candidate distribution converges toward a point mass at $1/2$. For $k = 3$, plurality becomes more complex: the winner need not be the closest to $1/2$ (e.g., with candidates at $1/3, 1/2$, and $2/3$). Nonetheless, we can still show that the candidate distribution converges to the center. To do so, we find an upper bound on $F_{3,t}(x)$ which goes to 0. The idea behind the proof is to enumerate cases where a candidate in an inner interval $(x, 1-x)$ wins and add up the probability of these cases, as a function of $F_{k,t-1}(x)$. By symmetry, we can transform a lower bound on the probability the winner is in $(x, 1-x)$ into an upper bound on $F_{3,t}(x)$. We then use induction to get a closed-form bound showing exponential convergence to the center.

Theorem 3. *Let $F_0 \in \mathcal{F}$. For all $x < 1/2$ and $t > 0$,*
$$F_{3,t}(x) \leq F_0(x) \cdot \left[3/4 + F_0(x)^2\right]^t.$$

For $k = 4$, we also derive an exponentially-shrinking upper bound on the CDF. However, the base of the exponential depends very strongly on x, increasing rapidly towards 1 near $1/2$, suggesting slower convergence to the center for $k = 4$ than for $k = 2$ or 3 (as seen in Figure 2). The proof follows a similar strategy as $k = 3$, but requires additional work to achieve a non-trivial bound.

Theorem 4. *Let $F_0 \in \mathcal{F}$. For all $x \in (1/3, 1/2)$ and $t \geq 0$,*
$$F_{4,t}(x) \leq F_0(x) \cdot \left[1 - 4(1/2 - F_0(x/3 + 1/3))^3\right]^t.$$

Note that $x/3 + 1/3$ is the point two-thirds of the way from x to $1/2$. As long as $F_0(x/3 + 1/3) < 1/2$, which is

true for any $x < 1/2$ if F_0 is positive near $1/2$, this result shows that the candidate distribution converges to the center.

In contrast to $k = 2, 3, 4$, we now show that for any larger k, the candidate distribution does not converge to the center. The proof is based on the following observation.

Lemma 1. *If all candidates are in $(1/4, 3/4)$, then only the left- or rightmost candidate can win with uniform voters.*

Proof. If all candidates are in $(1/4, 3/4)$, any candidate between two others gets vote share less than $1/4$, since no two candidates are distance $\geq 1/2$ apart; but the left- and rightmost candidates each get vote share $> 1/4$. □

Intuitively, if the candidate distribution starts converging to the center, then all candidates will likely be inside $(1/4, 3/4)$, at which point only the most extreme candidates can win. When k is sufficiently large, the left- and rightmost candidates are likely on opposite sides and farther from $1/2$ than the average candidate. This results in a centrifugal force preventing further progress towards the center. Formally, we prove the following non-convergence result for $k \geq 5$.

Theorem 5. *Let $F_0 \in \mathcal{F}$. For any $k \geq 5$, there exists some $x < 1/2$ such that $\lim_{t \to \infty} F_{k,t}(x) \neq 0$.*

Replicator Dynamics With Noise

We now show that our results still hold in approximate forms if some of candidates deviate from the winner-copying heuristic and instead position themselves randomly.

Definition 2. Given an initial candidate distribution F_0, candidate count k, and noise $\epsilon \in (0, 1]$, the *replicator dynamics for candidate positioning with ϵ-uniform noise* (under plurality with uniform 1-Euclidean voters) are, for all $t > 0$,

$$F_{k,t}^\epsilon(x) = \Pr(\text{Plur}(X_{1,t}^\epsilon, \ldots, X_{k,t}^\epsilon) \leq x), \quad (3)$$

$$X_{i,t}^\epsilon \sim \begin{cases} \text{Uniform}(0,1) & \text{w.p. } \epsilon, \\ F_{k,t-1}^\epsilon & \text{w.p. } 1 - \epsilon. \end{cases}$$

As in the noiseless case, we show that the candidate distribution converges to the center under the dynamics with ϵ-uniform noise for $k = 2, 3, 4$ but do not for $k \geq 5$. However, since ϵ-uniform noise introduces non-central candidates at every t, we need to relax the convergence requirement. The idea behind our notion of approximate convergence is that if we make the noise sufficiently small, then the distribution should get arbitrarily close to a point mass at $1/2$. That is, the CDF at any point $x < 1/2$ eventually goes below any positive threshold c, for any sufficiently small $\epsilon > 0$.

Definition 3. Let $F_0 \in \mathcal{F}$. The candidate distribution *approximately converges to the center* under the replicator dynamics with ϵ-uniform noise if for all $x \in [0, 1/2)$ and $c > 0$, there exists some $\epsilon_{\max} > 0$ such that if $\epsilon \in (0, \epsilon_{\max}]$, then $\limsup_{t \to \infty} F_{k,t}^\epsilon(x) < c$.

We then have the following analogue of our main result.

Theorem 6. *Let $F_0 \in \mathcal{F}$. For $k \in \{2, 3, 4\}$, the candidate distribution approximately converges to the center under replicator dynamics with ϵ-uniform noise. In contrast, for all $k \geq 5$, the candidate distribution does not approximately converge to the center.*

This result follows from an exact characterization of the limiting distribution for $k = 2$ (Theorem 7) and bounds for $k = 3$ (Theorem 8) and $k = 4$ (Theorem 9), as before. These results are stated below. Their proofs start with similar case analysis to the noiseless case, although now accounting for uniformly random candidates. We then analyze the convergence of the resulting iterated maps for $F_{k,t}^\epsilon(x)$. The non-convergence proof is also similar to the noiseless case.

Theorem 7. *Let $F_0 \in \mathcal{F}$. For any $\epsilon \in (0,1)$ and $x \in [0, 1/2]$ with ϵ-uniform noise,*

$$\lim_{t \to \infty} F_{2,t}^\epsilon(x) = \frac{1 - 4x\epsilon(1-\epsilon) - \sqrt{1 - 8\epsilon x(1-\epsilon)}}{4(1-\epsilon)^2} \leq \epsilon.$$

Theorem 8. *Let $F_0 \in \mathcal{F}$. For any $\epsilon \in (0, 1/3)$ and $x \in [0, 1/2]$, $\limsup_{t \to \infty} F_{3,t}^\epsilon(x) \leq 1.5\epsilon$.*

Theorem 9. *Let $F_0 \in \mathcal{F}$. For any $\epsilon \in (0,1]$ and $x \in (1/3, 1/2)$, let $\beta = 1/2 - \epsilon(x/3 + 1/3) - (1 - \epsilon)\max\{x/3 + 1/3, F_0(x/3 + 1/3)\}$. Then $\beta \in (0, 1/2]$ and $\limsup_{t \to \infty} F_{4,t}^\epsilon(x) \leq \frac{1}{8\beta^3}\epsilon$.*

A Result for $k \geq 5$ With No Extreme Candidates

In the previous sections, our results for $k \geq 5$ have been negative, showing the candidate distribution does not converge to the center, but without indicating what the distribution converges to instead. Simulations show a tendency towards two clusters (recall Figure 2), but we have not theoretically characterized the limiting distribution in general for $k \geq 5$. However, Lemma 1 allows us to analyze the dynamics in the special case that F_0 has no extreme candidates, with support only on $(1/4, 3/4)$. In this setting, the dynamics are much simpler, as only the left- and rightmost candidates can win. The same type of argument we used before for $k \geq 5$ then provides a positive result, showing that the candidate distribution converges to one with zero mass in an interval around $1/2$. In contrast, our central convergence results for $k \in \{2, 3, 4\}$ still hold in this special case.

Theorem 10. *Suppose $F_0 \in \mathcal{F}$ is supported on $(1/4, 3/4)$. Let $\ell = [1 - \sqrt{3/7}]/2 = 0.17\ldots$. For $k \geq 5$ and $x \in (F_0^{-1}(\ell), 1/2)$, $\lim_{t \to \infty} F_{k,t}(x) = 1/2$.*

When F_0 is Uniform$(1/4, 3/4)$, $F_0^{-1}(\ell) < 0.34$, so Theorem 10 implies that as $t \to \infty$, the candidate density in $[0.34, 0.66]$ goes to 0 for $k \geq 5$.

Simulations

Having established our main theoretical results, we demonstrate them in simulation and explore several extensions. To do so, we use Monte Carlo sampling, simulating a large number of elections per generation (100,000) and using the winners to approximate $F_{k,t}$. We initialize F_0 to be uniform. With this setup, we observe some effects due purely to sampling, such as oscillations due to small asymmetries in the Monte Carlo samples. To preserve the symmetry of the model, we use a trick we term *enhanced symmetry*, mirroring each copied position across $1/2$ with probability $1/2$.

First, recall Figure 2, which shows 50 layered simulation runs for $k = 2, \ldots, 7$ with enhanced symmetry. The candidate distributions evolve as we would expect from our theory: rapid convergence to the center for $k = 2$ and 3, slow convergence for $k = 4$, and non-convergence to the center for $k \geq 5$. Interestingly, $k = 5, 6$, and 7 show a tendency towards two point masses at $1/4$ and $3/4$. In the extended version, we show that these patterns are robust to different simulation choices, including with only 50 elections per generation, no enhanced symmetry, and larger candidate counts up to $k = 50$.

Variants of the Replicator Dynamics

We now demonstrate in simulation that the qualitative picture provided by our results from Theorem 1 is robust to different specifications of the model. At a high level, our model of candidate positioning consists of the following components: (1) a fixed voter distribution, (2) a subset of previous candidate positions which new candidates imitate, and (3) a rule for sampling from those previous positions. In the basic model, the voter distribution is uniform, the imitated positions are plurality winners from the previous generation, and the sampling rule chooses uniformly from those winners. Adding ϵ-uniform noise modifies the sampling rule to sometimes pick uniformly random positions. In simulation, we explore natural variations of each of these modeling components (see the extended version for formal definitions).

Non-uniform voters. Our simulations indicate the same convergence dichotomy for various symmetric unimodal and bimodal voter distributions. In Figure 4, we show simulation runs with the unimodal distribution Beta$(2, 2)$.

Memory. In this variant, candidates sample from winner positions in any of the last m generations. In Figure 4, we see that adding $m = 2$ generations of memory has little impact (the results for $m = 3$ were very similar).

Perturbation noise. With *perturbation noise*, each candidate slightly deviates from the position they copy, as if their imitation is imperfect. We add Gaussian noise with mean 0 and variance σ^2 to each copied position. Figure 4 shows that the candidate distribution with a small amount of perturbation noise ($\sigma^2 = 0.005$) converges to the center for $k = 2, 3, 4$ but does not for $k \geq 5$. However, with sufficient noise, we found that higher values of k can form a single central cluster. Additionally, for $k \geq 6$ the behavior varies significantly across runs without enhanced symmetry. We observed phenomena such as cluster drift, divergence, and extinction, particularly for higher values of k.

Variable candidate counts. In this variant, we allow elections in each generation to have a mixture of several candidate counts, where candidates copy from winner positions across all k in the previous generation. We find that our results interpolate smoothly in this setting: when most elections have fewer than five candidates, we see convergence to the center, but not when most elections have $k \geq 5$. Figure 4 shows results from an equal mixture of the listed candidate counts in each generation, with 50,000 elections per k.
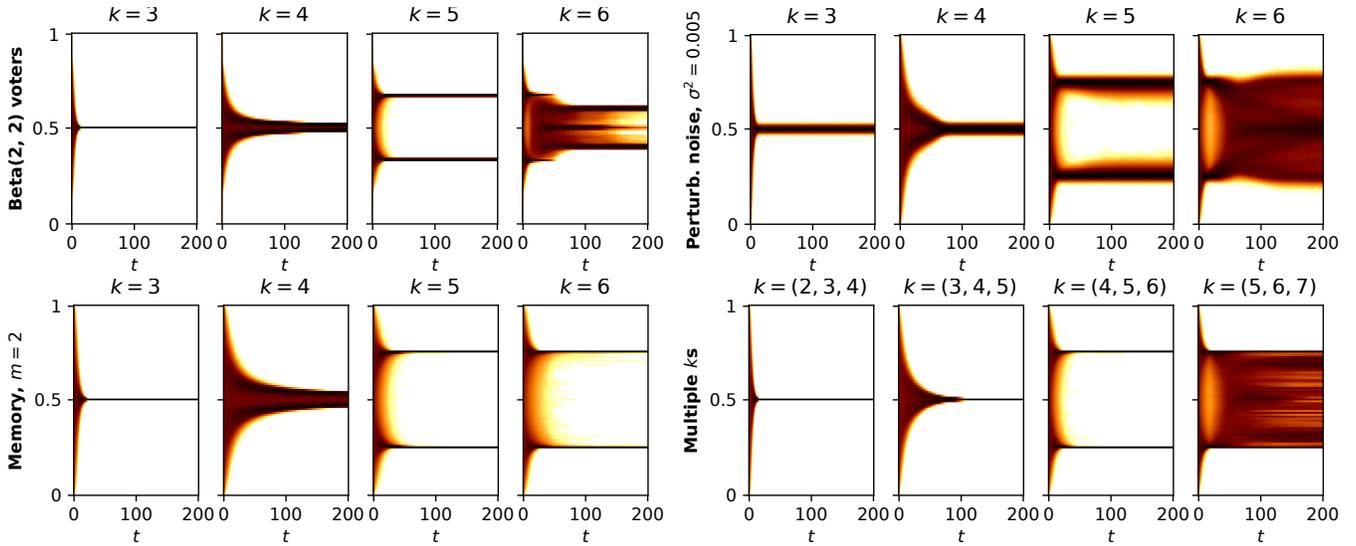
Figure 4: Variants of the replicator dynamics. Each plot shows 50 trials with no enhanced symmetry. Left column, top to bottom: unimodal Beta-distributed voters and 2 generations of memory. Right column, top to bottom: perturbation noise with $\sigma^2 = 0.005$ and variable candidate counts. Like the plain model, these variants converge to the center for $k < 5$. See the extended version for additional variant plots.

Top-h copying. Finally, we let candidates copy any position in the top-h highest vote shares in the previous generation, rather than only winning positions (i.e., $h = 1$). In simulation, top-h copying is the only variant which strays from the dichotomy we establish in Theorem 1. For $k = 3, 4$ and $h = 2$, the candidate distribution did not converge to the center, instead forming two clusters.

Relation to Nash Equilibria of One-Shot Games

We now examine the relationship between our model and prior work on one-shot strategic candidate positioning. With 1-Euclidean uniform voters, the Hotelling–Downs equilibrium has both candidates at $1/2$—which is also the attracting distribution of the replicator dynamics with $k = 2$. Indeed, Nash equilibria (NEs) of one-shot games are fixed points of the corresponding replicator dynamics (Hofbauer and Sigmund 2003), but replicator dynamics fixed points may not be NEs. We can see this in our setting since a distribution F is a (symmetric, mixed-strategy) NE if no strategy does better against F than sampling from F, while F is a fixed point of the replicator dynamics if no strategy *drawn from* F does better against F than sampling from F. For F with full support, symmetric mixed-strategy NEs and replicator dynamics fixed points coincide (Bauer, Broom, and Alonso 2019). However, NEs can be unstable under the dynamics, or their basins of attraction may be negligible.

Before analyzing NEs, we need to address what happens when multiple candidates occupy the same point—we call these *positional ties*. We have avoided this issue by using atomless candidate distributions, as such ties then occur with probability 0. One option for handling positional ties is to suppose that candidates fail to position themselves at *exactly*

the same point, with infinitesimal jitter determining a left–right order. Alternatively, we can suppose that candidates are in fact precisely at the same point, forcing voters to make an arbitrary choice between them (the rule in Cox (1987)).

Definition 4. Suppose multiple candidates occupy the same point. Under *left–right tie-breaking*, one of them (chosen u.a.r.) receives the vote share left of that point, while another (also u.a.r.) receives the vote share to the right. Under *equal split tie-breaking*, all candidates at a point share the vote share allocated to that point equally. Equivalently, voters randomly choose between equidistant candidates.

Given these positional tie-breaking rules, we prove several results showing how static analysis of NEs yields more fragile conclusions than the asymptotic behavior of the replicator dynamics. We focus on two types of equilibria: (1) symmetric mixed-strategy NEs (SMSNEs), since these relate to fixed points of the replicator dynamics; and (2) pure-strategy NEs (PSNEs), since these are the focus of classical candidate positioning analyses. We first show that there are multiple SMSNEs in the one-shot candidate positioning game, but they are often unstable or have tiny basins of attraction under the dynamics. In contrast, as we have seen in theory and simulation, the replicator dynamics behave in qualitatively similar ways under a range of specifications. Then, we show that PSNEs are sensitive to the choice of positional tie-breaking rule: we arrive at very different conclusions if we adopt left–right versus equal split tie-breaking. In contrast, the positional tie-breaking rule is irrelevant to the dynamics with atomless candidate distributions.

Symmetric Mixed-Strategy Nash Equilibria
Since SMSNEs are a subset of the replicator dynamics fixed points, we might hope to understand the dynamics by ana-

lyzing SMSNEs of the game where candidates seek to maximize their win probability. However, we find that there are multiple SMSNEs and they can have trivial basins of attraction. For instance, every candidate at $1/2$ is a SMSNE and a replicator dynamics fixed point (with left–right tie-breaking[1]). But as we proved, for $k \geq 5$ all symmetric atomless initial distributions do not converge to the center. On the other hand, if we allow initial distributions with atoms and the mass at $1/2$ is sufficiently high, the candidate distribution does indeed approach the all-at-$1/2$ SMSNE.

Theorem 11. *Suppose F_0 places probability mass p at $1/2$. For any $k \geq 2$, there is some $p_k^* < 1$ such that if $p > p_k^*$, the candidate distribution converges to a point mass at $1/2$ under the replicator dynamics with left–right tie-breaking.*

There is another family of SMSNEs where each candidate picks between the points x and $1 - x$ (for $x \in (1/4, 1/2)$).

Theorem 12. *With $k \geq 4$ and left–right tie-breaking, for any $x \in (1/4, 1/2)$, the strategy where each candidate picks uniformly at random between x and $1 - x$ is a SMSNE.*

Just as with the all-at-1/2 equilibrium, this SMSNE is not indicative of the typical behavior of the replicator dynamics. However, for distributions with atoms, the candidate distribution can converge to this type of equilibrium.

Theorem 13. *Suppose F_0 places probability mass p at x and at $1 - x$, for $1/4 < x < 1/2$. For $k \geq 5$, there is some $p_k^* < 1/2$ such that if $p > p_k^*$, the candidate dsn. converges to point masses at x and $1 - x$ under the replicator dynamics.*

These results show the existence of many SMSNEs that do not tell us how the dynamics typically behave.

Positional Tie-Breaking and PSNEs

We now demonstrate how focusing on static equilibria can yield results very sensitive to tie-breaking rules. Cox (1987) extends the Hotelling–Downs analysis to more than two candidates, characterizing PSNEs of a one-shot candidate positioning game—crucially, with equal split tie-breaking. With uniform voters and $k \geq 3$ candidates, Cox proves that there is no PSNE for odd k and that the only PSNE for even k has evenly-spaced pairs of candidates at $1/k, 3/k, \dots, (k-1)/k$. Clearly, this analysis makes very different predictions than our model. However, we show that Cox's results depend strongly on equal split tie-breaking. If we instead use left–right tie-breaking, the picture is very different. In particular, all candidates at $1/2$ is then a PSNE *for all* k: any deviant loses with certainty to the center candidate who captures the opposite side of the vote. Left–right tie-breaking also introduces many additional PSNEs. (Note that the objective we assume for candidates is equivalent to Cox's.)

Theorem 14. *Suppose candidates maximize their plurality win probability, then the vote margin against their strongest*

competitor, then second-strongest, etc. These are (some[2] of the) PSNEs with uniform voters and left–right tie-breaking:

1. *Any $k \geq 2$: all k candidates at $1/2$.*
2. *Any $k \geq 4$: for any $x \in (1/4, 1/2)$, $\lfloor k/2 \rfloor$ candidates at x, $\lfloor k/2 \rfloor$ at $1 - x$, and one (for odd k) at x or $1 - x$.*
3. *Any $k \geq 5$: $\lfloor (k-1)/2 \rfloor$ cands. at $1/4$, $\lfloor (k-1)/2 \rfloor$ at $3/4$, one at $1/2$, and one (for even k) at $1/4$ or $3/4$.*
4. *Even k: Cox's equilibrium; two candidates at each of the points $1/k, 3/k, \dots, (k-1)/k$.*

Thus, the qualitative conclusions we arrive by examining NEs are very different from Cox's if we make a different reasonable assumption. These results show how analyzing NEs provides a brittle picture of candidate positioning, yielding results that are sensitive to tie-breaking. Even SMSNEs, which are closely related to replicator dynamics fixed points, fail to reveal the typical behavior of the dynamics.

Discussion

We introduced a replicator dynamics model of one-dimensional candidate positioning in plurality elections based on simple heuristic inspired by bounded rationality. Our theoretical results show that the candidates converge to the center when there are at most four candidates per election, but diverge when there are five or more candidates per election. Simulations confirm that this pattern is robust to a large range of model variations. We highlight links between replicator dynamics and Nash equilibria, with our analysis of dynamics yielding more robust results. Overall, we reveal a plausible mechanism for the emergence of two parties in Duverger's Law: enough candidates imitating successful policies naturally converge to two ideological clusters.

Many open questions remain in the analysis of our model. The foremost is a characterization of the asymptotic candidate distribution for $k \geq 5$, although this may be challenging given the complex behavior we observe in high-k simulations. An even larger challenge is posed by expanding beyond symmetric and atomless initial candidate distributions to distributions which have points masses or are asymmetric. As we saw in Theorem 13, allowing atoms means there are infinitely many attracting distributions for $k \geq 5$, so the task becomes one of cataloguing all of the possible long-run candidate distributions and their basins of attraction. Theoretical results for our model variants would be interesting, such as characterizing which mixtures of candidate counts k lead to convergence to the center, or conditions on voter distributions that result in central convergence for $k \leq 4$.

While we explored several model variations in simulation, there are many more than can possibly be covered in a single paper. Additional variations of particular interest include policy-motivated candidates, strategic voters, probabilistic voters, and higher-dimensional preferences. Another natural direction would be to explore voting systems other than plurality, like instant runoff; Condorcet methods are trivial under our one-dimensional replicator dynamics, since the candidate closest to the median voter always wins, but might exhibit more complex behavior in higher dimensions.

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[1]In this subsection, we use left–right tie-breaking, as it yields equilibria closer to the typical behavior of the replicator dynamics; e.g., we will see that with equal split tie-breaking, all candidates at $1/2$ is only a SMSNE for $k = 2$—not $k = 3$ or 4, where we know the candidate distribution also converges to $1/2$.

[2]In the extended version, we show that for $k \leq 5$, this list of PSNEs is exhaustive; for $k > 6$, there may be others.

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