

Proportional Budget Allocations: Towards a Systematization

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Abstract

We contribute to the programme of lifting proportionality axioms from the multi-winner voting setting to participatory budgeting. We define novel proportionality axioms for participatory budgeting and test them on known proportionality-driven rules such as Phragmén and Rule X. We investigate logical implications among old and new axioms and provide a systematic overview of proportionality criteria in participatory budgeting.

1 Introduction

First introduced in Porto Alegre, Brazil in 1988, Participatory budgeting (PB) is a democratic budgeting practice in which citizens are consulted, through some voting method, on how to best allocate a given budget to public projects. The practice is attracting increasing attention from both democracy practitioners worldwide and researchers, among others within computational social choice [Aziz and Shah, 2021].

The axiomatic study of PB has formulated a number of criteria for the desirable behaviour of participatory budgeting methods, or PB rules. Special attention has been dedicated to forms of ‘fairness’ or ‘proportionality’. Intuitively, one may want a PB rule to output a division of the available budget over the projects that ‘reflects’ divisions in the voters’ preferences. A variety of proportionality axioms has been proposed in recent literature, and this paper provides a first systematization of the axiomatic landscape of proportionality in PB and its special case of multi-winner voting (MWV, [Faliszewski *et al.*, 2017]), or committee selection, that is, a PB setting where all projects (referred to as candidates) have identical cost.

State of the art. The current understanding of proportionality in PB is rooted in MWV [Skowron *et al.*, 2017; Peters, 2018]. A key fairness axiom in MWV is *justified representation* (JR) [Aziz *et al.*, 2017]. In short, JR requires that if a large enough group of voters agrees about a candidate, there is at least one candidate in the chosen committee that at least one of the group members approves. Proportionality requirements have been added with the axioms of *extended justified representation* (EJR) [Aziz *et al.*, 2017] and *proportional justified representation* (PJR) [Sánchez-Fernández *et al.*, 2017], following the intuition that if a larger group agrees

about more candidates, they should be represented by more candidates in the winning committee. Aziz *et al.* [2018] generalize these concepts from MWV to PB within an approval voting framework. A related fairness concept is the *core* [Fain *et al.*, 2016]. A set of projects, or *bundle*, is a core bundle if there is no subset of agents who can afford a different bundle (with their own share of the total budget) where every agent in that subset gets more utility than in the chosen bundle.

In [Peters and Skowron, 2020], two MWV rules considered to be proportional—Proportional Approval Voting (PAV) and Phragmén’s rule (or simply Phragmén)—are analysed, and shown to guarantee different types of proportionality. PAV induces a fair distribution of *welfare*, so every group of agents gets a utility proportional to its size, while Phragmén can be seen as inducing a fair distribution of *power*: the influence of a group of agents is proportional to its size. The authors introduce two new proportionality axioms: *priceability* and *laminar proportionality* (LP), and a new rule, Rule X, that is similar to both PAV and Phragmén, but satisfies both new axioms and the aforementioned EJR. Finally, work has started exploring the logical relations between the proportionality axioms proposed in the literature. For instance, Peters *et al.* [2021] show that, in MWV, the core implies EJR, PJR, and JR. Moving to PB, research has focused on assessing the extent to which the above MWV axioms and rules can be meaningfully generalised to PB. Pierczyński *et al.* [2021] generalise Rule X and EJR to PB, and show that, even in this context, Rule X satisfies EJR.¹ The PB variant of Rule X is also shown to satisfy an approximation of the core and the axiom of priceability for PB. PAV is generalised to PB too, and shown to fail EJR without the unit-cost assumption.

Contribution. We make two contributions. First, we complete the study of proportionality for Phragmén and Rule X in the PB setting with respect to the axioms mentioned above (Table 3). To do so, we propose novel generalizations of PJR and LP (Definitions 7 and 11) from the MWV to the PB setting. Second, we provide an overview of the logical relations between proportionality axioms in MWV and PB, establishing several novel results (see Figure 1). The resulting picture contributes to a systematization of how proportionality is interpreted in PB. Only selected proofs are provided.

¹Pierczyński *et al.* [2021] call Rule X in the PB setting ‘method of equal shares’ (MES). For simplicity we stick to the term Rule X.

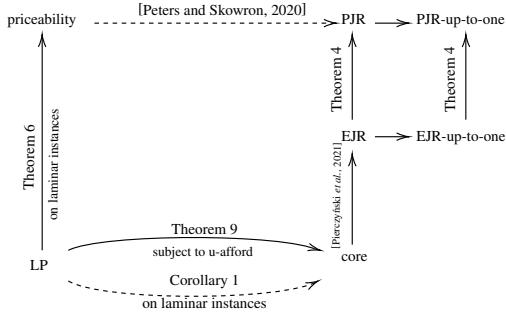


Figure 1: Relations among proportionality axioms in PB. Dashed lines indicate relations that only hold in MWV. Arrows are labelled either by our results or the paper where they have been proven. Some of the implications only hold under certain conditions or restrictions: *laminar instances* (Definition 10), and *unanimity affordability* (u-afford, Definition 12). Transitive arrows are omitted. Absence of arrows denotes the existence of a counterexample.

2 Preliminaries

2.1 The Participatory Budgeting (PB) Problem

We denote the set of projects (or candidates) by $C = \{c_1, c_2, \dots, c_m\}$ and the set of voters by $N = \{v_1, v_2, \dots, v_n\}$. Each voter i comes with a function u_i assigning a utility to all projects. In the fully general setting of utility-based PB, $u_i : C \rightarrow [0, 1]$. In contrast, in the special case of approval-based PB, the utility function is restricted to two values: $u_i : C \rightarrow \{0, 1\}$, determining voter i 's approval set $A_i = \{c \in C : u_i(c) = 1\}$. The utility of a set of projects $T \subseteq C$ for a set of voters $S \subseteq N$ is defined additively as: $u_S(T) = \sum_{i \in S} \sum_{c \in T} u_i(c)$. A *profile* P is a vector of the utility functions of all voters: $P = (u_1, \dots, u_n)$. If utilities are approval-based we denote with $C(P)$ the set of all projects occurring in the approval sets in P . A function cost: $C \rightarrow \mathbb{Q}_+$ assigns a cost to every project. The cost of a set of projects T is given by $\text{cost}(T) = \sum_{c \in T} \text{cost}(c)$. The total budget is denoted by l . If l is not mentioned, it is equal to 1. Hence, an election instance (also called a *PB-instance*) $E = (N, C, \text{cost}, P, l)$ consists of a set of voters N , a set of projects C , a cost function, a profile P , and a budget l . If all else is clear in context, we abbreviate this to $E = (P, l)$. A *voting rule* \mathcal{R} maps an election instance E to a winning bundle W . A PB-instance where for all $i \in N$ $u_i : C \rightarrow \{0, 1\}$ is called *approval-PB-instance*. The special case of an approval-PB-instance in which all projects have the same cost is called a *MWV-instance*. In such MWV setting, we refer to a bundle as a *committee*.

2.2 Voting Rules

We focus on three proportionality-inspired rules: (sequential) Phragmén, proportional approval voting (PAV) and Rule X. Below, we recall the generalizations of Phragmén, PAV, and Rule X to PB introduced by Pierczyński *et al.* [2021].

Phragmén Every voter gets currency continuously at the rate of one unit of currency per unit of time. At the first moment t when there is a group of voters S who all approve a not-yet-selected project c , and who together have $\text{cost}(c)$ units of currency, the rule adds c to the bundle and asks the

project	cost	utilities				Winning bundles		
		v_1	v_2	v_3	v_4	Phragmén	PAV	Rule X
c_1	0.4	1	0.7	0.1	0			✓
c_2	0.3	0.3	0.4	0	0.4	✓	✓	
c_3	0.7	0.1	0.2	0.4	0.4		✓	
c_4	0.35	0	0.4	0.2	1	✓		✓

Table 1: The profile used in Examples 1 and 2. Each column contains the utilities per project of a voter. The budget $l = 1$.

voters from S to pay the cost of c (i.e., the rule resets the balance of each voter from S), while the others keep their so-far earned money. The process stops when it would select a project which would overshoot the budget.

PAV The winning bundle W of PAV is the bundle with $\text{cost}(W) \leq l$ that maximises the score PAV-score(W) = $\sum_{i \in N} \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{|W \cap A_i|}\right)$.

Rule X The rule starts by giving each voter an equal fraction of the budget. In case of a budget of 1, each of the n voters gets $\frac{1}{n}$ unit of currency. We start with an empty bundle $W = \emptyset$ and sequentially add projects to W . To add a project c to W , the voters have to pay for c . Write $p_i(c)$ for the amount that voter i pays for c ; we will need that $\sum_{i \in N} p_i(c) = \text{cost}(c)$. Let $p_i(W) = \sum_{c \in W} p_i(c) \leq \frac{1}{n}$ be the total amount voter i has paid so far. For $\rho \geq 0$, we say that a project $c \notin W$ is ρ -affordable if $\sum_{i \in N} \min(\frac{1}{n} - p_i(W), u_i(c) \cdot \rho) = \text{cost}(c)$. The rule iteratively selects a project $c \notin W$ that is ρ -affordable for a minimum ρ . Individual payments are given by $p_i(c) = \min(\frac{1}{n} - p_i(W), u_i(c) \cdot \rho)$. If no project is ρ -affordable for any ρ , Rule X terminates and returns W .

Remark 1. Note that Phragmén and PAV work with approval-based while Rule X with utility-based ballots. In what follows, when discussing the first two rules, we will therefore presuppose approval-PB-instances. Notice also that all three rules are non-resolute.

Example 1. Consider the profile in Table 1. To get the approval profile needed for Phragmén and PAV, we binarize the utility function using a threshold of 0.3: voters approve a project when it yields utility of at least 0.3. Approved projects are shaded. Phragmén will first select c_2 at time $t = 0.1$, which leaves voter v_3 with 0.1 units of currency. Then at $t = 0.275$, v_2 and v_4 can together buy c_4 , which leaves v_1 with 0.175 and v_3 with 0.275. After adding c_4 , the rule ends since both remaining projects are not affordable and outputs $W = \{c_2, c_4\}$. For PAV, we compute the PAV-score of four sets: PAV-score($\{c_1, c_4\}$) = 3.5, PAV-score($\{c_1, c_2\}$) = 4, PAV-score($\{c_2, c_3\}$) = 4.5, and PAV-score($\{c_2, c_4\}$) = 4 (clearly, smaller sets have a lower score and larger sets are not affordable). Hence, PAV will select $W = \{c_2, c_3\}$. Rule X starts by giving every voter $\frac{1}{4}$ unit of currency. First, c_4 is ρ -affordable for $\rho = \frac{0.35}{1.6} \approx 0.219$, then, for $\rho = \frac{0.4}{1.8} \approx 0.222$, c_1 is ρ -affordable. After selecting c_1 , the sum of the remaining amounts of all voters is 0.25, so the other projects are not ρ -affordable for any ρ . Hence, Rule X returns $W = \{c_1, c_4\}$.

2.3 Known Proportionality Axioms

We recall axioms for PB from Pierczyński *et al.* [2021] that generalize known MWV axioms. We start with the axioms of

core and extended justified representation (EJR).

Definition 1 (Core). *For a given PB-instance $E = (N, C, \text{cost}, P, l)$, a bundle W is in the core if for every $S \subseteq N$ and $T \subseteq C$ with $|S| \geq \frac{\text{cost}(T)}{l} \cdot n$ there exists $i \in S$ such that $u_i(W) \geq u_i(T)$. A voting rule \mathcal{R} satisfies the core property if for each PB-instance E the winning bundle $\mathcal{R}(E)$ is in the core.*

To define EJR, we first introduce (α, T) -cohesiveness:

Definition 2 $((\alpha, T)$ -cohesiveness). *A group of voters S is (α, T) -cohesive for $\alpha : C \rightarrow [0, 1]$ and $T \subseteq C$, if $|S| \geq \frac{\text{cost}(T)}{l} \cdot n$ and if it holds that $u_i(c) \geq \alpha(c)$ for every voter $i \in S$ and each project $c \in T$.*

Definition 3 (Extended justified representation). *A rule \mathcal{R} satisfies extended justified representation (EJR) if for each PB-instance E and each (α, T) -cohesive group of voters S , there is a voter $i \in S$ such that $u_i(\mathcal{R}(E)) \geq \sum_{c \in T} \alpha(c)$.*

In Pierczyński *et al.* [2021], a weakening of EJR is considered, to which we refer as *EJR-up-to-one*.

Definition 4 (Extended justified representation up to one project). *A rule \mathcal{R} satisfies extended justified representation up to one project (EJR-up-to-one) if for each PB-instance E and each (α, T) -cohesive group of voters S , there is a voter $i \in S$ such that $u_i(\mathcal{R}(E)) \geq \sum_{c \in T} \alpha(c)$ or for some $a \in C$ it holds that $u_i(\mathcal{R}(E) \cup \{a\}) > \sum_{c \in T} \alpha(c)$.*

In MWV, Definitions 3 and 4 are equivalent:

Proposition 1. *Def. 3 and Def. 4 are equivalent in MWV-instances.*

We now turn to priceability, for which the notion of price system needs to be introduced first.

Definition 5 (Price systems). *A price system is a pair $\mathbf{ps} = (b, (p_i)_{i \in N})$ where $b \geq 1$ is the initial budget, and for each voter $i \in N$, there is a payment function $p_i : C \rightarrow \mathbb{R}$ such that (1) a voter can only pay for projects she gets at least some utility from: if $u_i(c) = 0$, then $p_i(c) = 0$ for each $i \in N$ and $c \in C$, and (2) each voter can spend the same budget of $\frac{b}{n}$ units of currency: $\sum_{c \in C} p_i(c) \leq \frac{b}{n}$ for each $i \in N$.*

Definition 6 (Priceability). *A rule \mathcal{R} satisfies priceability (is priceable) if for each PB-instance E , there exists a price system $\mathbf{ps} = (b, (p_i)_{i \in N})$ that supports $\mathcal{R}(E)$, that is: (1) for each $c \in \mathcal{R}(E)$, the sum of the payments for c equals its price, i.e., $\sum_{i \in N} p_i(c) = \text{cost}(c)$; (2) no project outside of the winning bundle gets any payment, i.e., for all $c \notin \mathcal{R}(E)$, $\sum_{i \in N} p_i(c) = 0$; (3) there exists no non-selected project whose supporters in total have a remaining unspent budget of more than its cost, i.e., for all $c \notin \mathcal{R}(E)$,*

$$\sum_{i \in N \text{ s.t. } u_i(c) > 0} \left(\frac{b}{n} - \sum_{c' \in \mathcal{R}(E)} p_i(c') \right) \leq \text{cost}(c).$$

Remark 2. *The above properties are defined for PB. Throughout the paper, when needing to refer to the MWV specialization of a PB axiom, we will assume the axiom is defined on MWV-instances. We say that a rule satisfies an axiom in MWV-instances if, for all MWV-instances, the rule's winning bundle satisfies the axiom. Observe also that, since Phragmén and PAV are defined on approval-PB-instances, when we assess whether they satisfy an axiom, we consider the axiom only with respect to approval-PB-instances.*

3 Two Novel Axioms for PB

3.1 Proportional Justified Representation in PB

Sánchez-Fernández *et al.* [2017] define proportional justified representation for MWV (we refer to it as MWV-PJR here). We generalize this axiom to PB based on the generalisation of EJR provided by Pierczyński *et al.* [2021]. Two steps are involved: dropping the unit-cost assumption, and allowing arbitrary utilities instead of just approval ones.

Definition 7 (Proportional justified representation). *A rule \mathcal{R} satisfies proportional justified representation (PJR) if for each PB-instance E and (α, T) -cohesive group of voters S ,*

$$\sum_{c \in \mathcal{R}(E)} (\max_{i \in S} u_i(c)) \geq \sum_{c \in T} \alpha(c). \quad (1)$$

The intuition is that, in the winning bundle, for each cohesive group S (cohesive in that S agrees to a certain degree about the set of projects T) there should be enough projects to which at least one voter in S assigns enough utility.

Example 2. *Consider the profile in Table 1, with the bundle $W = \{c_2, c_3\}$ as selected by PAV. Now consider the group $S = \{v_1, v_2\}$. Probably both voters in S are happy that c_2 is selected, but they would both get more utility from c_1 than from the selected c_2 or c_3 . Also, if each of them would get their share of the total budget ($\frac{1}{4}$), they could together afford the set $T = \{c_1\}$. Intuitively then, W is not a fair bundle considering voters v_1 and v_2 . Let us look at it more formally. S is (α, T) -cohesive for $\alpha(c_1) = 0.7$ ($\alpha(c_2)$, $\alpha(c_3)$ and $\alpha(c_4)$ are arbitrary) since the voters in S can afford T with their share of the budget, and for both of them $u(c_1) \geq \alpha(c_1)$. However, Equation 1 is not satisfied: $\sum_{c \in W} (\max_{i \in S} u_i(c)) = 0.4 + 0.2 = 0.6$, while $\sum_{c \in T} \alpha(c) = 0.7$. This shows that in the given election instance, the bundle W does not satisfy PJR.*

Our definition of PJR for PB is rather different from its MWV variant. It requires some work to show that the proposed definition reduces to the definition of MWV-PJR under unit-cost assumption and approval preferences. First of all, let us recall the definition of MWV-PJR:

Definition 8 (PJR for MWV [Sánchez-Fernández *et al.*, 2017]). *An approval based voting rule \mathcal{R} satisfies PJR for MWV (MWV-PJR) if for every ballot profile P and committee size k , the rule outputs a committee $W = \mathcal{R}(P, k)$ s.t.: for every $\ell \leq k$ and every ℓ -cohesive set of voters $S \subseteq N$, it holds that $|W \cap (\cup_{i \in S} A_i)| \geq \ell$, where a set S is ℓ -cohesive if $|S| \geq \ell \cdot \frac{n}{k}$ and $|\cap_{i \in S} A_i| \geq \ell$.*

We will need the following lemma:

Lemma 1. *Let $E = (N, C, \text{cost}, P, l)$ be an approval-PB-instance where for all projects c , $\text{cost}(c) = \frac{1}{k}$. Then: (a) for given $\alpha : C \rightarrow [0, 1]$ and $T \subseteq C$, any group of voters $S \subseteq N$ that is (α, T) -cohesive is also ℓ -cohesive for $\ell = |T'|$ with $T' = \{c \in T : \alpha(c) > 0\}$; and (b) for every group S that is ℓ -cohesive there are $T \subseteq C$ with $|T| = \ell$ and $\alpha : C \rightarrow [0, 1]$ with $\alpha(c) = 1$ for all $c \in C$, such that S is (α, T) -cohesive.*

Theorem 1. *MWV-PJR and PJR are equivalent in MWV-instances.*

Proof Sketch. We show that on MWV-instances, Definitions 7 and 8 are equivalent. $\boxed{\text{PJR} \Rightarrow \text{MWV-PJR}}$ Assume that a rule \mathcal{R} satisfies PJR, and take an arbitrary MWV-instance E . Because E satisfies the assumptions of unit-cost and approval based voting, the fact that \mathcal{R} satisfies PJR boils down to the following: for all $S, \alpha : C \rightarrow [0, 1]$, and $T \subseteq C$ with $|S| \geq |T| \cdot \frac{n}{k}$ and for which $u_i(c) \geq \alpha(c)$ for all $i \in S$ and for all $c \in T$, it is the case that $|\mathcal{R}(E) \cap (\cup_{i \in S} A_i)| \geq \sum_{c \in T} \alpha(c)$. Take arbitrary $S \subseteq N$ and $\ell \leq k$ and suppose that S is ℓ -cohesive. According to Lemma 1(b), there are $T \subseteq C$ with $|T| = \ell$ and $\alpha : C \rightarrow [0, 1]$ with $\alpha(c) = 1$ for all $c \in C$, such that S is (α, T) -cohesive. Because \mathcal{R} satisfies PJR, this implies that $|\mathcal{R}(E) \cap (\cup_{i \in S} A_i)| \geq \sum_{c \in T} \alpha(c)$. However, because of our choice of T and α , we know that $\sum_{c \in T} \alpha(c) = |T| = \ell$, so $|\mathcal{R}(E) \cap (\cup_{i \in S} A_i)| \geq \ell$, which shows that \mathcal{R} satisfies MWV-PJR. $\boxed{\text{MWV-PJR} \Rightarrow \text{PJR}}$ Assume that a rule \mathcal{R} satisfies MWV-PJR. Take arbitrary MWV-instance E , and suppose that a group S is (α, T) -cohesive. Then according to Lemma 1(a), when we take $T' = \{c \in T : \alpha(c) > 0\}$, S is ℓ -cohesive for $\ell = |T'|$. Because R satisfies MWV-PJR, it follows that $|\mathcal{R}(E) \cap (\cup_{i \in S} A_i)| \geq \ell = |T'|$. By definition of T' , $|T'| \geq \sum_{c \in T'} \alpha(c) = \sum_{c \in T} \alpha(c)$, so $|\mathcal{R}(E) \cap (\cup_{i \in S} A_i)| \geq \sum_{c \in T} \alpha(c)$ as desired. \square

Remark 3. To our knowledge, besides Definition 7, the only generalization of MWV-PJR to PB is from Aziz et al. [2018]. They define an axiom called Strong-BPJR-L (where L stands for the budget limit, to which we refer here as ℓ) that requires the following: For a budget l , a bundle W satisfies Strong-BPJR-L if for all $\ell \in [1, l]$ there does not exist a set of voters $S \subseteq N$ with $|S| \geq \ell \frac{n}{l}$, such that $\text{cost}(\cap_{i \in S} A_i) \geq \ell$ but $\text{cost}((\cup_{i \in S} A_i) \cap W) < \ell$. It is possible to generalise this definition further to allow arbitrary utilities instead of approval votes. However, note that the requirement in this definition is not that for every ℓ -cohesive S the utility of the projects they all approve that are selected is at least ℓ , but rather the cost of this set of projects. Although this is indeed a generalisation of MWV-PJR, as is shown by Aziz et al. [2018], we consider that the aim of PJR is to ensure a certain level of utility for every group of voters, rather than a certain cost. Definition 7 is equivalent to Strong-BPJR-L when assuming that a project's cost is directly proportional to a voter's utility from it.

Like for EJR (Definition 4), we can add an up-to-one-project condition to PJR:

Definition 9 (Proportional justified representation up to one project). A rule \mathcal{R} satisfies proportional justified representation up to one project (PJR-up-to-one) if for each PB-instance E and each (α, T) -cohesive group of voters S , $\sum_{c \in \mathcal{R}(E)} (\max_{i \in S} u_i(c)) \geq \sum_{c \in T} \alpha(c)$ or for some $a \in C$ it holds that $\sum_{c \in \mathcal{R}(E) \cup \{a\}} (\max_{i \in S} u_i(c)) > \sum_{c \in T} \alpha(c)$.

3.2 Laminar Proportionality in PB

The basic idea of LP for MWV [Peters and Skowron, 2020] is that if we know about a strict separation between different parties, we can divide the chosen projects proportionally over the parties. We generalize this notion to PB in the approval voting setting by taking the budget l instead of the bundle size k , and by using the cost of each project instead of unit-cost.

$c_3, 3$		
$c_2, 3$		$c_5, 4$
$c_1, 2$		$c_4, 2$
	$c_6, 1$	
v_1	v_2	v_3

Table 2: Example of a laminar proportional bundle W (shaded) in a laminar election instance. Each column represents the approval set of a voter (written beneath it), and each box shows a project with its cost. E.g., voter v_2 approves c_6, c_1, c_2 , and c_3 and cost(c_5) = 4.

Definition 10 (Laminar PB-instances). An approval-PB-instance (P, l) is laminar if either: (1) P is unanimous and $\text{cost}(C(P)) \geq l$; (2) there is $c \in C(P)$ such that $c \in A_i$ for all $A_i \in P$, the profile P_{-c} (i.e., P once we remove c) is not unanimous and instance $(P_{-c}, l - \text{cost}(c))$ is laminar (with $P_{-c} = (A_1 \setminus \{c\}, \dots, A_n \setminus \{c\})$); or (3) There are two laminar PB-instances (P_1, l_1) and (P_2, l_2) with $C(P_1) \cap C(P_2) = \emptyset$ and $|P_1| \cdot l_1 = |P_2| \cdot l_2$ such that $P = P_1 + P_2$ and $l = l_1 + l_2$.

Example 3. The instance P in Table 2 associated with the budget $l = 10$ is laminar. The instance P_1 with v_1 and v_2 and projects c_1, c_2 , and c_3 with limit $l_1 = 6$ satisfies the first item of Definition 10, as does the instance P_2 with only voter v_3 and projects c_4 and c_5 , and limit $l_2 = 3$. Those two instances can be added by Definition 10, item 3, since $|P_1| \cdot l_2 = 2 \cdot 3 = 1 \cdot 6 = |P_2| \cdot l_1$. Then c_6 can be added by Definition 10, item 2, to get P with limit $l = 6 + 3 + \text{cost}(c_6) = 10$.

Definition 11 (Laminar proportionality). A rule \mathcal{R} satisfies laminar proportionality (LP) if for every laminar PB-instance $E = (P, l)$, $\mathcal{R}(E) = W$ where W is a laminar proportional bundle, i.e.: (1) if P is unanimous, then $W \subseteq C(P)$ (if everyone agrees, then part of the projects they agree on is chosen); (2) if there is a unanimously approved project c s.t. $(P_{-c}, l - \text{cost}(c))$ is laminar, then $W = W' \cup \{c\}$ where W' is laminar proportional for $(P_{-c}, l - \text{cost}(c))$; or (3) If P is the sum of laminar PB-instances (P_1, l_1) and (P_2, l_2) , then $W = W_1 \cup W_2$ where W_1 is laminar proportional for (P_1, l_1) and W_2 is laminar proportional for (P_2, l_2) .

It is trivial that in case of unit-cost and budget k , these definitions are equivalent to the corresponding MWV definitions.

Example 4. Elaborating on Example 3, bundle $W = \{c_1, c_2, c_4, c_6\}$ (grey in Table 2) is laminar proportional in that instance with a budget of $l = 10$. In (P_1, l_1) , $\{c_1, c_2\}$ is laminar proportional, as is $\{c_4\}$ in (P_2, l_2) . Hence, $\{c_1, c_2, c_4\}$ is laminar proportional in $(P_1 + P_2, l_1 + l_2)$, and $\{c_1, c_2, c_4, c_6\}$ is laminar proportional in (P, l) .

4 Proportionality Properties of Rules

Table 3 summarizes the findings of this section.

The literature already provides several results about PAV: it does not satisfy the core, priceability, or LP in MWV (and hence not in PB either). In MWV-instances PAV satisfies PJR, but it does not in PB:

Proposition 2. PAV does not satisfy PJR.

We turn now to our analysis of Phragmén and Rule X.

Proposition 3. Phragmén satisfies PJR.

	PAV	Phragmén	Rule X	
	MWV	PB	MWV	PB
core	\times [Aziz <i>et al.</i> , 2017]	\times [Brill <i>et al.</i> , 2017]	\times [Peters and Skowron, 2020]	\times [Pierczyński <i>et al.</i> , 2021]
EJR	\checkmark [Aziz <i>et al.</i> , 2017]	\times [Pierczyński <i>et al.</i> , 2021]	\times [Brill <i>et al.</i> , 2017]	\checkmark [Peters and Skowron, 2020]
PJR	\checkmark [Sánchez-Fernández <i>et al.</i> , 2017]	\times (Prop. 2)	\checkmark [Brill <i>et al.</i> , 2017]	\checkmark (Prop. 3)
Price	\times [Peters and Skowron, 2020]	\times [Peters and Skowron, 2020]	\checkmark [Peters and Skowron, 2020]	\checkmark [Peters and Skowron, 2020]
LP	\times [Peters and Skowron, 2020]	\checkmark [Peters and Skowron, 2020]	\times (Prop. 5)	\checkmark [Peters and Skowron, 2020]
			\checkmark [Peters and Skowron, 2020]	\times (Prop. 5)

Table 3: Three rules and the properties they satisfy: \checkmark indicates satisfaction, \times failure. Shaded entries indicate new results, references to the corresponding propositions or literature are included for each entry. Recall that PAV and Phragmén are assessed on approval-PB-instances.

Proof. Assume towards a contradiction that there exist a group of voters $S \subseteq N$, a set of projects $T \subseteq C$, and a function $\alpha : C \rightarrow [0, 1]$ such that S is (α, T) -cohesive, and for this S , α , and T , the winning bundle W of Phragmén does not contain enough projects that voters from S like enough: $\sum_{c \in W} (\max_{i \in S} u_i(c)) < \sum_{c \in T} \alpha(c)$. Note that in approval-PB-instances this boils down to $|W \cap \bigcup_{i \in S} A_i| < \sum_{c \in T} \alpha(c)$. Because of (α, T) -cohesiveness, for every voter $i \in S$ and each project $c \in T$, $u_i(c) \geq \alpha(c)$, so either $\alpha(c) = 0$ or $c \in A_i$, and therefore $\sum_{c \in T} \alpha(c) \leq |T \cap \bigcap_{i \in S} A_i|$. We write T' for $T \cap \bigcap_{i \in S} A_i$, and W' for $W \cap \bigcup_{i \in S} A_i$. Hence, $|W'| < |T'| \leq |T|$. Let t be the moment when the rule stops: a project c is reached that would overshoot the budget. Clearly, $\text{cost}(W) + \text{cost}(c) > 1$, but $\text{cost}(W) \leq 1$. Let x be the amount of virtual money earned by all voters so far, so

$$t \cdot n = x = \text{cost}(W) + \text{cost}(c) + y, \quad (2)$$

where $y \geq 0$ is the money that non-supporters of c have earned in the meantime. Because $|W'| < |T'|$, there must be some project in T that is not in W . The voters in S together have earned $\frac{x}{n} \cdot |S|$, and because S is (α, T) -cohesive, $|S| \geq \text{cost}(T) \cdot n$, so

$$\frac{x}{n} \cdot |S| \geq \text{cost}(T) \cdot n \cdot \frac{x}{n} = \text{cost}(T) \cdot x. \quad (3)$$

From (2) and the fact that $\text{cost}(W) + \text{cost}(c) > 1$, it follows that $x > 1 + y$ (and $x > 1$). From (3), it follows that the voters in S have earned enough together at time t to buy all projects from T (and therefore from T'), but have not done so. Hence, either they have also paid for projects not in T' , or, if they only spent their money on projects in T' , c must be in T' , i.e. they do have the virtual money to buy T' but it would overshoot the budget. In the first case, for every project not in T' that members of S pay for, $|W'|$ grows by one (since they can only pay for projects they approve). In order to keep $|W'| < |T'|$, the mean amount of money they have paid at time t for such a project must be greater than the mean cost of projects in T' . Otherwise, the number of projects they would pay for (that they approve of and that are selected) would exceed the number of projects in T' . However, for each project not in T' that voters from S pay for, they should (as a group) pay less than the cost of any project from T' not yet selected. Otherwise, they would have paid earlier for a cheaper project from T' . This is a contradiction. Hence, the voters from S only spent their money on projects from T' , and $c \in T'$. Let us assume that c is the last project from T' that is not yet selected.² Because the rule stops exactly when c can be paid by its supporters, we know that at

²If there are more projects from T' not yet selected, we get that $x \leq \text{cost}(T') \cdot \frac{n}{|S|}$, so $x \leq 1$ still holds.

that point in time, the voters in S have earned exactly $\text{cost}(T')$ units of money, so $t \cdot |S| = \text{cost}(T')$. Hence, the total amount of money earned at time t is $x = t \cdot n = \text{cost}(T') \cdot \frac{n}{|S|}$. Because S is (α, T) -cohesive, $\text{cost}(T') \leq \text{cost}(T) \leq \frac{|S|}{n}$, so $x = \text{cost}(T') \cdot \frac{n}{|S|} \leq \frac{|S|}{n} \cdot \frac{n}{|S|} = 1$. However, we also had that $x > 1 + y > 1$. Contradiction. \square

Proposition 4. Phragmén satisfies priceability.

Peters and Skowron [2020] show that Phragmén does not satisfy EJR in MWV- and, therefore, PB-instances. Since the core implies EJR, Phragmén does not satisfy the core neither in MWV- nor in PB-instances.

Proposition 5. Phragmén and Rule X do not satisfy LP.

Proof sketch. LP requires any affordable unanimously approved project to be selected, but Phragmén and Rule X do not necessarily select those if their cost is high enough compared to the other projects. \square

Proposition 6. Rule X satisfies PJR-up-to-one.

Proof. Rule X satisfies EJR-up-to-one [Pierczyński *et al.*, 2021]. Theorem 4 will show that EJR-up-to-one implies PJR-up-to-one. Hence Rule X also satisfies PJR-up-to-one. \square

5 Relations Between Axioms

We study the logical relationships among the axioms we introduced and report on the results recapitulated in Figure 1.

5.1 Priceability, PJR, EJR, and the Core

We start by showing that PJR, EJR, and the core do not imply priceability in MWV-instances, even in laminar ones.

Theorem 2. *In laminar MWV-instances there exist bundles that satisfy PJR, EJR, or are in the core, but are not priceable.*

Since laminar MWV-instances are a specific type of MWV-instances, which are a specific type of PB-instances, the same result holds for MWV and PB.

In MWV-instances, every priceable bundle satisfies PJR, as is shown by [Peters and Skowron, 2020, Prop. 1]. This raises the question whether this relation is also present in the PB setting. We show now that this is not the case.

Theorem 3. *There are priceable bundles not satisfying PJR.*

Proof sketch. PJR is based on the utility of the voters being higher than some threshold $\alpha(c)$, while priceability only discriminates between utilities of 0 and utilities above zero. It is therefore possible to construct a PB-instance with a bundle that is priceable but does not satisfy PJR. \square

From this result, it follows that priceability neither implies EJR nor the core. Priceability would otherwise imply PJR, since the core implies EJR, and EJR implies PJR (Theorem 4). These results hold even for MWV.

Theorem 4. *EJR(-up-to-one) implies PJR(-up-to-one).*

5.2 Laminar Proportionality and Priceability

We first show that in MWV-instances, and therefore also in PB-instances, priceability does not imply LP.

Theorem 5. *In laminar MWV-instances there exist priceable bundles that are not laminar proportional.*

However it is worth reporting that for a class of price systems (called balanced systems) we are able to prove that the implication goes through. Furthermore, LP implies priceability on laminar election instances even in the PB setting.

Theorem 6. *LP implies priceability in laminar PB-instances.*

Proof sketch. By induction on the structure of laminar PB-instances. We show that for every bundle W that is laminar proportional for a laminar PB-instance (P, l) , there exists a price system $\mathbf{ps} = (b, (p_i)_{i \in N})$ where $b = \text{cost}(W)$.

Consider three cases. The first is the basis of the induction. If P is unanimous with $\text{cost}(C(P)) \geq l$ and W is laminar proportional for (P, l) (with $\text{cost}(W) \leq l$), then $W \subseteq C(P)$, so voters can divide their budget over W . With an initial $b = \text{cost}(W)$, all and only the projects in W can be bought.

In the second case, a unanimously approved project $c \in W$ such that $W' = W \setminus \{c\}$ is laminar proportional. By the inductive hypothesis, we know that there exists a price system \mathbf{ps}' with initial budget $b' = \text{cost}(W')$. Because c is unanimously approved, all voters could pay for c . We know that in \mathbf{ps}' , there was no project not in W' that was affordable to its supporters. If every voter got $\frac{\text{cost}(c)}{n}$ more budget, to spend entirely on c , c would get funded and no voter would have more unspent budget than before. Also, the initial budget of every voter is now $\frac{b'}{n} + \frac{\text{cost}(c)}{n} = \frac{\text{cost}(W') + \text{cost}(c)}{n} = \frac{\text{cost}(W)}{n} = \frac{b}{n}$ units of money, and the initial budget is $b = \text{cost}(W)$ and all the individual payment functions stay the same. Because for every project c in W' the sum of the individual payments was equal to $\text{cost}(c)$, this is also the case for every project in W .

In the third case, the profile consists of two laminar profiles: $P = P_1 + P_2$ and $l = l_1 + l_2$, where W_1 and W_2 are laminar proportional for respectively (P_1, l_1) , (P_2, l_2) . Take $W = W_1 \cup W_2$, which is by definition laminar proportional for (P, l) . By the inductive hypothesis, there exist price systems $\mathbf{ps}_1 = (b_1, \{p_{1,i}\}_{i \in N})$ and $\mathbf{ps}_2 = (b_2, \{p_{2,i}\}_{i \in N})$ with initial budgets $b_1 = \text{cost}(W_1)$ and $b_2 = \text{cost}(W_2)$, and that $|P_1| \cdot l_2 = |P_2| \cdot l_1$. We can now define a price system \mathbf{ps} that supports W as follows: $\mathbf{ps} = (b, (p_i)_{i \in N})$ with $b = \text{cost}(W) = b_1 + b_2$, and for all voters $i \in N$, $p_i(c) = p'_{1,i}(c) + p'_{2,i}(c)$, where $p'_{1,i}$ and $p'_{2,i}$ are extended versions of respectively $p_{1,i}$ and $p_{2,i}$ that yield zero for the projects that those are not defined for. It is easy to check that \mathbf{ps} supports W according to the criteria from Def. 6. \square

5.3 Laminar Proportionality and the Core

Theorem 7. *In laminar MWV-instances, there exist bundles that satisfy PJR, EJR, or are in the core, but do not satisfy LP.*

Theorem 8. *There exist laminar proportional bundles that do not satisfy PJR, EJR, or are not in the core.*

Theorem 8 holds because of instances where there is one relatively expensive unanimously approved project, that is included in any laminar proportional bundle, but where there are many cheap projects that can be satisfactory enough for a group of voters. We show now that under certain restrictions, laminar proportional bundles are in the core (and hence also satisfy EJR and PJR). To do that we define a property of bundles which we call *unanimity-affordability* (u-affordability):

Definition 12. *A bundle T is u-affordable (shortly, u-afford) w.r.t. instance (P, l) whenever for any unanimously approved project $c \in C(P)$ there exists $t \in T$ s.t. $\text{cost}(t) \geq \text{cost}(c)$.*

Since $\text{cost}(c) \geq \text{cost}(c)$, the definition is also satisfied if $c \in T$. We will show that in laminar PB-instances, a laminar proportional bundle satisfies the core if it is *subject to u-afford*, that is, if u-afford holds for the bundle inductively on the structure of the instance.

Theorem 9. *In laminar PB-instances, laminar proportional bundles subject to u-afford satisfy the core.*

Proof sketch. The proof is by induction on laminar PB instances. Recall that a set of projects is in the core if for every group of voters $S \subseteq N$ and set of projects $T \subseteq C$ such that S can afford T with their share of the budget there is a voter $i \in S$ such that $u_i(W) \geq u_i(T)$. The proof then shows that if for any unanimously approved project c in W either c is part of T , or there is some project in T that costs at least as much as c , then there exists a voter $i \in S$ such that $u_i(W) \geq u_i(T)$. The interesting case concerns the step with a unanimous candidate c : any group S that could possibly block the core with bundle T should all prefer T over W or not be able to buy W . This is impossible because by u-afford T contains a project with a larger cost than c . \square

Since the core implies EJR and PJR (as showed by Pierczyński *et al.* [2021] and Theorem 4), any laminar profile also satisfies EJR and PJR under the same restrictions. Note that in MWV-instances there always is a project in T that has cost of at least $\text{cost}(c)$, so our restriction applies. Hence, there, laminar bundles are in the core, and so satisfy EJR and PJR.

Corollary 1. *In laminar MWV-instances, LP implies PJR, EJR, and the core.*

6 Conclusions and Future Work

Our study is a contribution towards the systematization of the axiomatic landscape of proportionality in PB (Figure 1, Table 3). With respect to proportionality-inspired rules, we showed that priceability and PB generalizations of PJR and LP do not discriminate between Phragmén and Rule X (unlike EJR).

We focused on the basic PB framework and on cardinal and approval-based utilities. Extending the study to richer PB settings (e.g., diversity constraints [Bredereck *et al.*, 2018; Lang and Skowron, 2018; Aziz, 2019], project groups [Jain *et al.*, 2021], negative attitudes [Talmon and Page, 2021], or resource types [Aziz and Shah, 2021]), or the yet more general ordinal preferences setting [Aziz and Lee, 2021], are natural avenues of future research.

Acknowledgments

The authors are greatly indebted to Jan Maly, who pointed out a mistake in an earlier version of the paper, and to the anonymous reviewers of IJCAI/ECAI'22 for many helpful comments. Zoé Christoff acknowledges support from the project Social Networks and Democracy (VENI project number VI.Veni.201F.032) financed by the Netherlands Organisation for Scientific Research (NWO). Davide Grossi acknowledges support by the Hybrid Intelligence Center, a 10-year program funded by the Dutch Ministry of Education, Culture and Science through the Netherlands Organisation for Scientific Research (NWO).

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