# Multi-Leader Congestion Games with an Adversary

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We study a multi-leader single-follower congestion game where multiple users choose subsets out of a given set of resources and an adversary attacks the resources with maximum loads, causing additional costs for the users. We first show that the resulting strategic game among the leaders admits a pure Nash equilibrium if the users' strategy-spaces are given by matroids and the resource costs are linear and identical. However, equilibria may fail to exist already when strategy spaces are not matroids or the linear cost coefficients are not identical. We therefore consider approximate equilibria for the case that each user chooses one resource and the resource costs are linear but non-identical. Our main result establishes that $K \approx 1.1974$ is the smallest possible factor such that the existence of a $K$-approximate equilibrium is guaranteed for all instances of the game. We also provide a polynomial time algorithm for computing an $\alpha$-approximate equilibrium with the smallest possible factor $\alpha \leq K$ for a given instance, in particular finding an exact equilibrium if one exists.

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## 1 Introduction

*Congestion games* serve as a fundamental model for the competition over scarce, congestion-sensitive resources: Users (or players) choose subsets of a common resource set and the cost a user experiences depends on the utilization of the chosen resources. Applications range from network routing, wireless spectrum sharing, cloud computing, scheduling, facility location, and many more. From a theoretical point of view, congestion games are conceptually simple and have appealing properties, for instance, a pure Nash equilibrium is guaranteed to exist by the famous result of Rosenthal [54]. This combination of nice theoretic behaviour and the ability to capture many aspects of practical applications has spurred the interest of the research community and led to many deep and influential results (see, e.g., [56, 16]).

In this paper, we study a model where a congestion game is complemented with an adversary subsequently attacking some of the resources after the players of the congestion game have chosen their strategies, causing additional costs for them. Specifically, we assume that the adversary is greedy, wanting to cause as much damage

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as possible, and, consequently, only attacks resources having largest load. For classic load balancing congestion games [10] or network congestion games [54], this type of adversary is very natural: Consider the scenario of clients each wanting to run a job on a computer server which can be selected from a set of available servers. In Denial-of-Service (DoS) attacks, it makes sense for the adversary to attack a server with maximum load, because this way, the attack reaches the largest number of clients and, ultimately, causes maximum damage. As another example, consider the scenario that users choose one of several entries in order to enter a sports arena. For this example, a promotional campaign who wants to reach as many people as possible will use an entry point with maximum load causing additional delays to the entry.

The model described above constitutes a *Stackelberg*, or *Leader-Follower* game—a game in which the players are partitioned into *leaders*, acting first, and *followers*, choosing their strategy only after the leaders' choices become apparent. Such hierarchical relationships appear in many real-world problems, for example, in applications from the security domain [34, 35, 59, 21, 55, 44, 4], pricing or toll setting problems [33, 37, 28, 27], supply chain and marketing management [29], voting scenarios [60], and many others. Due to this broad variety of applications, Stackelberg games have received considerable attention from scientists in artificial intelligence, operations research, theoretical computer science, economics, and other disciplines. While most works in the literature consider the case of a single leader, the case of *multiple leaders* (that occurs in our model) playing a simultaneous-move strategic game subject to one or more followers has received much less attention and only few results with respect to the existence and computational complexity of equilibria are known. Note that the multi-leader case applies to several scenarios, for example, in the analysis of deregulated electricity markets in which some of the large energy producers are the leaders and the smaller energy producers and independent system operators are the followers; see [39], or in the security context when multiple defenders protect common targets, see [38, 43, 21, 20]. One well-known obstacle in the analysis of multi-leader games—even in the realm of continuous formulations with convex action spaces for the leaders and single-valuedness of the followers' response—is the inherent non-convexity of the best-response correspondence that results in non-existence of pure Nash equilibria [36, 21].

Indeed, it turns out that pure Nash equilibria fail to exist for the class of games studied in this paper, already in very simple settings: Even in *symmetric singleton* congestion games where each player chooses exactly one of the resources, and linear congestion cost functions, one cannot guarantee existence of pure Nash equilibria among the leaders. (The adversary is assumed to use a fixed budget to attack one of the maximum load resources uniformly at random.) The observed non-existence of pure Nash equilibria motivates two major research questions that we study in this paper for congestion games with an adversary: Firstly, the analysis of conditions leading to the existence (or non-existence) of pure Nash equilibria. Secondly, in the absence of pure Nash equilibria, the study of alternative solution concepts. In this paper, we focus on *approximate* pure Nash equilibria where no unilateral deviation improves the cost of the deviating leader by more than a certain factor.

## 1.1 Our Results and Proof Techniques

In this paper, we study multi-leader congestion games with an adversary and assume linear congestion cost functions (for a discussion of the limitations of this and other assumptions, see Section 6.2). Regarding the existence of pure Nash equilibria for such multi-leader games, we show that if the linear cost coefficients of all resources are identical and the leaders' strategies are given by base sets of player-specific matroids, equilibria are guaranteed to exist. To prove this, we use a special lexicographical ordering on the resource load vectors, and, making use of the basis exchange property, show that a minimum corresponds to a pure Nash equilibrium. On the other hand, we observe that if the resource cost coefficients are non-identical, equilibria do not always exist, not even for symmetric singleton action spaces of the leaders (where each leader is allowed to choose exactly one of the available resources). This motivates the analysis of approximate pure Nash equilibria, where no

unilateral deviation improves the cost of the deviating leader by more than a factor $\alpha$, for some $\alpha \geq 1$. We analyze existence and efficient computation of approximate equilibria for the case that the underlying congestion game is a symmetric singleton congestion game (and the congestion costs are linear). Note that although the assumption of singleton strategy spaces is certainly restrictive, this class is still rich enough to model many applications [31], among them the described example of clients choosing servers and a subsequent DoS attack, and has attracted a lot of attention in the literature, for example, [19, 7, 12, 13, 11]. Moreover, the analysis of approximate equilibria in our model turns out to be challenging already for singleton strategies.

As our main result, we show that $\alpha = K \approx 1.1974$ is the smallest possible value of $\alpha$ such that the existence of an $\alpha$-approximate equilibrium can always be guaranteed, where $K$ is the unique solution of the equation $-x^3 + x^2/2 + 1 = 0$. For the proof, we give an efficient algorithm computing a $K$-approximate equilibrium. The basic approach is to start with an empty game, and add the players one after another, always placing them on a best-response resource. If the addition of a new player makes some of the already added players "unhappy", meaning that there is a unilateral deviation decreasing their cost by more than a factor $\alpha$, we let the unhappy players deviate one after another until all players are happy again, that is, an $\alpha$-approximate pure Nash equilibrium is reached for the subset of players already added. Only then the next player is added. By choosing the possible deviations carefully, we can show that this procedure terminates after a polynomial number of steps for $\alpha = K$, showing existence of $K$-approximate pure Nash equilibria and giving an efficient way of computing them. A similar approach has been used before to compute *exact* pure Nash equilibria for weighted congestion games [1, 47], but we are not aware of any results regarding approximate equilibria applying this technique. We furthermore provide an instance which does not admit an $\alpha$-approximate equilibrium for any $\alpha < K$. This shows that $\alpha = K$ is tight in the sense that it is the smallest possible value such that the existence of an $\alpha$-approximate equilibrium can be guaranteed for *all* instances of the introduced game.

However, for a specific given instance, better approximate equilibria might exist, that is, there may be an $\alpha$-approximate equilibrium with $\alpha < K$. We show how to compute efficiently, for a given instance, a *best* approximate equilibrium, that is, with smallest possible value of $\alpha$ such that an $\alpha$-approximate equilibrium exists. Note that this in particular implies that we can decide efficiently whether a given instance admits an exact pure Nash equilibrium, and in case of existence, we can also compute such an equilibrium. Our algorithm is based on a careful analysis of the structure of optimal approximate equilibria, which allows us to enumerate a polynomially-sized set of possible resource-load configurations.

## 1.2 Related Work

The games analyzed in this paper are an extension of classical congestion games. Different generalizations of congestion games are analyzed in the literature, including games with weighted players, player-specific cost functions [47], local effect games [40], and resource graph games [32, 26]. In resource graph games, as in our model, the cost of each resource depends on the whole vector of resource loads. Harks et al. [26] show that pure Nash equilibria in these games are guaranteed to exist for arbitrary strategy sets if and only if the dependence of the cost of a resource on the loads of other resources is linear and symmetric. In our model this dependence is non-linear, but also the strategy spaces are matroids or singletons and, hence, more restricted. For weighted congestion games, a pure Nash equilibrium may also fail to exist [25], but the existence and computation of approximate equilibria is fairly well understood [9, 8, 14, 22, 23].

The adversary model we consider can be seen as a particular model of a resource failure. Penn et al. [52, 53, 51] consider congestion games where resources fail with a certain probability and derive existence and efficient computation of pure Nash equilibria. In contrast to our work, players are allowed to choose *any* subset of the resources and the resource failure rate of a resource depends only on its own load. In our adversary model, the additional cost of a resource due to the adversary action depends on the load of *all* resources. Nickerl and

Torán [48] consider fixed resource failure probabilities but more general strategy spaces and show existence of pure Nash equilibria as well as hardness of computation. Wang et al. [57] analyze under which conditions the pure Nash equilibria of the original standard congestion game are preserved in a model with fixed resource failure probabilities. Bilò et al. [6] consider a network routing game where one edge is adversarially destroyed for each player. They study the existence and quality of a particular class of mixed strategies in this setting. Congestion games with *agent* failures are studied in [46]. The authors show that pure Nash equilibria always exist and study the price of anarchy for the case of identical and independent failure probabilities. Li et al. [41] consider resource *and* agent failures simultaneously for congestion games where the symmetric strategy sets are given by $k$-uniform matroids. They show the existence of pure Nash equilibria and analyze the quality for the case of identical players and identical resources. They also study a model with correlated resource failures (but no agent failures) and derive existence and efficient computation of pure Nash equilibria.

Babaioff et al. [3] consider a non-atomic congestion game with a malicious player controlling a certain amount of the flow with the incentive to maximize the average delay of the induced network congestion game. Malicious players in *atomic* congestion games are studied by Gairing [17], who assumes that each player has a fixed probability to be malicious. Congestion games in which the players are partly *altruistic* are considered, for example, by Hoefer and Skopalik [30]. They show that pure Nash equilibria are guaranteed to exist and can be computed efficiently for matroid congestion games and convex congestion cost functions. They also study the level of altruism that needs to be added in order to achieve certain behaviour, for example, that a social optimum coincides with a pure Nash equilibrium, for singleton congestion games. This is also studied by Apt and Schäfer [2] for linear congestion cost functions.

There are several works analyzing hierarchical games where in the lower level *non-atomic* users settle in a Wardrop equilibrium [58]. Marcotte [45] and Garing et al. [18] study an optimization problem where a single leader invests in network capacities with the goal of minimizing the sum of the total congestion costs and the costs for installing of capacities. Beckmann et al. [5] and Yang and Huang [61] study the problem of a single leader to set road tolls. Correa et al. [15] and Harks et al. [28] consider a game where multiple leaders set prices in order to maximize their own profits achieved in a network routing game. The prices that the leaders are allowed to choose are upper-bounded by price caps (leader-specific or equal for all leaders, respectively), and the two papers consider the three-level problem of a system designer who chooses the cap(s) in order to minimize total congestion. Finally, models where multiple leaders choose prices and capacities to maximize their individual profits in a network routing game are, for example, analyzed by Johari et al. [33], Liu et al. [42], and Harks and Schedel [27].

Castiglioni et al. [12] and Marchesi et al. [44] consider a model in which the players of the congestion game are partitioned into a (single) leader and multiple followers (but the leader's congestion cost functions may be different from the followers'). Depending on the structure of strategy spaces and congestion cost functions, they analyze the computational complexity of computing pure Nash equilibria. In particular, they find that efficient algorithms are only possible for singleton strategy spaces (unless $\mathsf{P} = \mathsf{NP}$), and derive such algorithms for singleton strategy spaces where either all followers have the same strategies [12], or the followers can be divided in "classes" having the same strategies [44].

## 2 The Model

For an integer $k \in \mathbb{Z}_{\geq 0}$, let $[k] := \{1, \ldots, k\}$. Let $N = [n]$ be a finite set of players (leaders) and $R = \{r_1, \ldots, r_m\}$ be a finite set of $m$ resources. For each player $i \in N$, the set of strategies available to player $i$ is a subset $X_i \subseteq 2^R$. We call $x = (x_1, \ldots, x_n)$ with $x_i \in X_i$ for all $i \in N$ a *strategy profile*, and $X = X_1 \times \cdots \times X_n$ the *strategy space*.

We use standard game theory notation; for a strategy profile $x \in X$, we write $x = (x_i, x_{-i})$ meaning that $x_i$ is the strategy that player $i$ plays in $x$ and $x_{-i}$ is the partial strategy of all players except $i$. Every strategy profile

$x = (x_1, \ldots, x_n) \in X$ induces a *load* or *congestion* on the resources given by $\ell_r(x) := |\{i \in N : r \in x_i\}|, r \in R$. We assume *linear* congestion cost functions $a_r \ell_r(x), r \in R$ with nonnegative coefficients $a_r \geq 0$. In classical congestion games, the private cost of player $i$ under strategy profile $x \in X$ is then defined as $\sum_{r \in x_i} a_r \ell_r(x)$. In the game studied in this paper, the players experience additional costs caused by the adversary that we describe now. The *adversary* (follower) has a budget of $B > 0$ that is used to attack one of the maximum-load resources uniformly at random. Define for a strategy profile $x \in X$ the maximum load as $M(x) := \max_{r \in R}\{\ell_r(x)\}$ and the resources with maximum load as $M^{-1}(x) := \arg\max_{r \in R}\{\ell_r(x)\}$. From the perspective of a player (leader) in the congestion game, the expected additional cost on resource $r \in R$ due the attack of the adversary is

$$\kappa_r^*(x) := \begin{cases} \frac{B}{|M^{-1}(x)|}, & \text{if } r \in M^{-1}(x), \\ 0, & \text{else.} \end{cases}$$

The multi-leader congestion game with an adversary that we study in this paper can be defined as the game $G = (N, R, X, \pi)$ in strategic form, where

$$\pi_i(x) := \sum_{r \in x_i} \left(a_r \ell_r(x) + \kappa_r^*(x)\right) \text{ for all } i \in N. \tag{1}$$

We furthermore define

$$c_r(x) := a_r \ell_r(x) + \kappa_r^*(x)$$

as the cost of resource $r \in R$ under strategy profile $x$ with the *congestion part* $a_r \ell_r(x)$, and the *adversary part* $\kappa_r^*(x)$. A strategy profile $x \in X$ is called a *pure Nash equilibrium (PNE)* of $G$ if

$$\pi_i(x) \leq \pi_i(y_i, x_{-i}) \text{ for all } i \in N \text{ and } y_i \in X_i.$$

As one of our first observations (see the next section), we find that pure Nash equilibria do not always exist for the introduced congestion games with an adversary. This motivates the analysis of approximate equilibria defined as follows.

Definition 1. *A strategy profile $x \in X$ is an $\alpha$-approximate pure Nash equilibrium ($\alpha$-PNE) of $G$ for some $\alpha \geq 1$, if*

$$\pi_i(x) \leq \alpha \cdot \pi_i(y_i, x_{-i}) \text{ for all } i \in N \text{ and } y_i \in X_i.$$

*A unilateral deviation which decreases the cost of the deviating player more than a factor of $\alpha$ is called an $\alpha$-improving deviation, or an $\alpha$-improving move.*

For $\alpha = 1$, we obtain the standard (exact) PNE. For general $\alpha \geq 1$, the interpretation is that no player can improve her cost by a unilateral deviation gaining more than a factor $\alpha$. We remark that, while one can similarly define *additively* approximate equilibria, no existence guarantees can be given for such equilibria for any additive constant due to the scale-invariance of the games studied in this article (see Section 6.1 for details). Furthermore, despite the fact that the existence of *mixed-strategy* equilibria is guaranteed, pure strategies are favoured in many cases since mixed strategies do not always have a meaningful interpretation (see, for example, the discussion on mixed strategies in Section 3.2 of [49]).

## 3 On the Existence of Pure Nash Equilibria

In this section, we analyze existence of pure Nash equilibria for the introduced congestion games with an adversary. As the main result, we show that pure Nash equilibria exist if the players' strategy spaces consist of (player-specific) base sets of matroids *and* all resource cost coefficients $a_r$, $r \in R$, are identical. However, we also find that if we relax at least one of these two conditions, that is, strategy spaces are not matroids or cost coefficients are different, one cannot guarantee existence of pure Nash equilibria anymore.

We start with the positive result about matroid congestion games and to this end, we need to introduce some notation. An ordered pair $(R, \mathcal{I})$ with $\mathcal{I} \subseteq 2^R$ is a *matroid* if the following three properties are satisfied:

(i) $\emptyset \in \mathcal{I}$;
(ii) If $x \subseteq y$ and $y \in \mathcal{I}$, then $x \in \mathcal{I}$;
(iii) If $x, y \in \mathcal{I}$ and $|x| < |y|$, then there exists $r \in y \setminus x$ such that $x \cup \{r\} \in \mathcal{I}$.

The sets in $\mathcal{I}$ are called *independent sets*. Specifically, the (inclusion-wise) *maximal* independent sets are the *bases* of the matroid $\mathcal{M} = (R, \mathcal{I})$. As an easy example, if $\mathcal{I}$ consists of the empty set together with all singletons $\mathcal{I} = \emptyset \cup \{\{r\}, r \in R\}$, then $(R, \mathcal{I})$ is a matroid and the base set is $\{\{r\} : r \in R\}$. For more information on matroids, see for instance the book by Oxley [50].

If in a congestion game, the players' strategies are bases of matroids, that is, for each $i \in N$ there exists a matroid $\mathcal{M}_i = (R, \mathcal{I}_i)$ such that $X_i$ equals the base set of $\mathcal{M}_i$, we call the game a *matroid* congestion game. In particular, singleton congestion games are matroid congestion games.

As the following theorem shows, existence of pure Nash equilibria can be guaranteed in matroid congestion games with an adversary if all cost coefficients are identical:

Theorem 2. *Let $X_i, i \in N$ correspond to the base sets of player-specific matroids $\mathcal{M}_i = (\mathcal{I}_i, R)$. If $a_r = a_{r'}$ holds for all $r, r' \in R$, then exact PNE do exist.*

*Proof.* We use the *sorted lexicographical order* of load vectors to construct a PNE. For two vectors $a, b \in \mathbb{R}^m$, denote by $\tilde{a}$ and $\tilde{b}$ the sorted vectors obtained by permuting the entries of $a, b$ in non-increasing order, that is, $\tilde{a}_1 \geq \tilde{a}_2 \geq \cdots \geq \tilde{a}_m$ and similarly for $\tilde{b}$. Then, $a$ is *strictly sorted lexicographically smaller* than $b$, written as $a <_{lex} b$, if there is an index $1 \leq j \leq m$ with $\tilde{a}_i = \tilde{b}_i$ for all $1 \leq i < j$ and $\tilde{a}_j < \tilde{b}_j$. The vector $a$ is *sorted lexicographically smaller* than $b$ (written as $a \leq_{lex} b$), if $a <_{lex} b$ or $\tilde{a} = \tilde{b}$.

Choose a strategy profile $x$ whose load vector $\ell(x)$ is sorted lexicographically minimal, that is, $\ell(x) \leq_{lex} \ell(y)$ for all $y \in X$. We argue that $x$ is a PNE. To see this, consider any alternative strategy $y_i \in X_i$ of some player $i$ and let $y := (y_i, x_{-i})$. By the basis exchange property (see, e.g., Lemma 1.2.2 in [50]), there is a bijection $\sigma : y_i \to x_i$ with $\sigma(r) = r$ for $r \in x_i \cap y_i$ and $y_i^r := (x_i \setminus \{\sigma(r)\}) \cup \{r\} \in X_i$ for all $r \in y_i$. Note that $\ell(x) \leq_{lex} \ell(y_i^r, x_{-i})$ by lexicographic minimality of $x$, which implies

$$\ell_{\sigma(r)}(x) \leq \ell_r(y_i^r, x_{-i}) = \ell_r(y) \tag{2}$$

for all $r \in y_i$.

Note also that $M(x) \leq M(y)$ by lexicographic minimality of $x$. We distinguish two cases:

- If $M(y) > M(x)$, then $M^{-1}(y) \subseteq y_i$ because $\ell_r(y) \leq \ell_r(x)$ for all $r \in R \setminus y_i$. Hence $\pi_i(y) = B + \sum_{r \in y_i} a_r \ell_r(y) \geq B + \sum_{r \in y_i} a_{\sigma(r)} \ell_{\sigma(r)}(x) \geq \pi_i(x)$, where we use (2) and the premise that $a_r = a_{r'}$ for all $r, r' \in R$.
- If $M(y) = M(x)$, then note that $r \in M^{-1}(y) \cap y_i$ for all $r$ with $\sigma(r) \in M^{-1}(x) \cap x_i$ by (2). Therefore $|M^{-1}(y) \cap y_i| \geq |M^{-1}(x) \cap x_i|$ and $M^{-1}(y) \setminus y_i \subseteq M^{-1}(x) \setminus x_i$, from which we conclude that $|M^{-1}(y) \cap y_i|(|M^{-1}(x) \cap x_i| + |M^{-1}(x) \setminus x_i|) \geq |M^{-1}(x) \cap x_i|(|M^{-1}(y) \cap y_i| + |M^{-1}(y) \setminus y_i|)$ and hence

$$\sum_{r \in y_i} \kappa_r^*(y) = \frac{|M^{-1}(y) \cap y_i|}{|M^{-1}(y)|} \cdot B \geq \frac{|M^{-1}(x) \cap x_i|}{|M^{-1}(x)|} \cdot B = \sum_{r \in x_i} \kappa_r^*(x).$$

  Using (2) and the premise that $a_r = a_{r'}$ for all $r, r' \in R$, we again obtain $\pi_i(y) = \sum_{r \in y_i} \kappa_r^*(y) + \sum_{r \in y_i} a_r \ell_r(y) \geq \sum_{r \in x_i} \kappa_r^*(x) + \sum_{r \in y_i} a_{\sigma(r)} \ell_{\sigma(r)}(x) = \pi_i(x)$.

We conclude that $\pi_i(y) \geq \pi_i(x)$ and hence a deviation to $y_i$ does not improve the cost for player $i$. □

In the remainder of this section, we will present various examples showing that matroid strategy spaces and identical cost coefficients are, in a sense, necessary for guaranteeing existence of pure Nash equilibria. All our examples live in the realm of *network congestion games on parallel paths*. Formally, in such a game, there is a network consisting of two nodes, a source $s$ and a sink $t$, together with $k \geq 1$ edge-disjoint parallel paths $p_1, \ldots, p_k$ from $s$ to $t$, where each path $p_i$ consists of $m_i \geq 1$ edges. If for each player $i \in N$, the strategy set $X_i = \{p_1, \ldots, p_k\}$, we speak of a network congestion game on parallel paths. In the context of network congestion games, we also use the terms resources and edges interchangeably.

The following lemma provides useful sufficient conditions for the non-existence of pure Nash equilibria in (specific) instances with underlying network congestion game on parallel paths.

Lemma 3. *Consider a congestion game with an adversary, where the congestion game is a network congestion game on $k \geq 1$ parallel paths $p_1, \ldots, p_k$ with numbers of edges $1 \leq m_1 < \cdots < m_k$. Assume further that all edge cost coefficients are identical and strictly positive $a_1 = \cdots = a_m =: a > 0$. Then:*

*If there is no pure Nash equilibrium $x$ such that the induced path loads $\ell_p(x) := |\{i \in N : x_i = p\}|$ satisfy $\ell_{p_1}(x) \geq \cdots \geq \ell_{p_k}(x)$, then the instance does not have* any *pure Nash equilibrium.*

*Proof.* Assume, by contradiction, that there is a pure Nash equilibrium $x$. By assumption, there exist paths $p_a, p_b$ with $m_a < m_b$ but $\ell_{p_a}(x) < \ell_{p_b}(x)$. We show that if one player, say player $i$, deviates from $p_b$ to $p_a$, this move improves the cost of player $i$, contradicting the fact that $x$ is an equilibrium.

Note first that the *congestion part* of player $i$'s cost clearly decreases:

$$\sum_{r \in p_b} a_r \ell_r(x) = \sum_{r \in p_b} a\ell_{p_b}(x) = m_b \cdot a\ell_{p_b}(x) > m_a \cdot a\ell_{p_b}(x) \geq m_a \cdot a(\ell_{p_a}(x) + 1) = \sum_{r \in p_a} a_r \ell_r(p_a, x_{-i}).$$

We show in the following that the adversary part does not increase, that is, $\sum_{r \in p_b} \kappa_r^*(x) \geq \sum_{r \in p_a} \kappa_r^*(p_a, x_{-i})$, completing the proof.

Trivially, we have $0 \leq \sum_{r \in p_a} \kappa_r^*(p_a, x_{-i}) \leq B$. Consequently, the statement holds if $\sum_{r \in p_b} \kappa_r^*(x) = B$ or $\sum_{r \in p_a} \kappa_r^*(p_a, x_{-i}) = 0$. Let us thus assume that both equations are not fulfilled, that is, $\sum_{r \in p_b} \kappa_r^*(x) < B$ and $\sum_{r \in p_a} \kappa_r^*(p_a, x_{-i}) > 0$. The first inequality implies that there exists a path having maximum load in $x$ which is different from $p_b$ (and $p_a$). The second inequality shows that $p_a$ is a maximum load path after one player deviated from $p_b$ to $p_a$. Taking both inequalities together shows that $\ell_{p_a}(x) = \ell_{p_b}(x) - 1 = M(x) - 1$ is the only possible remaining case, where $M(x) = \max\{r \in R : \ell_r(x)\}$ denotes the maximum load in $x$. In this case, we have that

$$\sum_{r \in p_b} \kappa_r^*(x) = m_b \cdot \frac{B}{|M^{-1}(x)|} > m_a \cdot \frac{B}{|M^{-1}(x)| - m_b + m_a} = \sum_{r \in p_a} \kappa_r^*(p_a, x_{-i}),$$

where the inequality uses that $|M^{-1}(x)| > m_b > m_a > 0$ and $|M^{-1}(x)|$ denotes the number of maximum load resources in $x$. □

The lemma that we just proved becomes useful in the following example. It demonstrates that if the cost coefficients are identical, but players' strategy spaces are not bases of matroids anymore, pure Nash equilibria may fail to exist.

Example 4. *Consider the network displayed in Figure 1. We have three parallel paths $p_1, p_2, p_3$ with $m_1 = 1, m_2 = 4$, and $m_3 = 9$ many edges. All linear edge cost coefficients are equal to $a = 1$. Finally, assume that the number of players is $n = 5$, where the strategy set of each player $i$ is $X_i = \{p_1, p_2, p_3\}$, and the adversary's budget equals $B = 7$. We proceed to show that the corresponding multi-leader congestion game with an adversary does not admit a pure Nash equilibrium. To this end, we prove that for each strategy profile, there exists a player who can decrease her experienced cost by a unilateral deviation.*

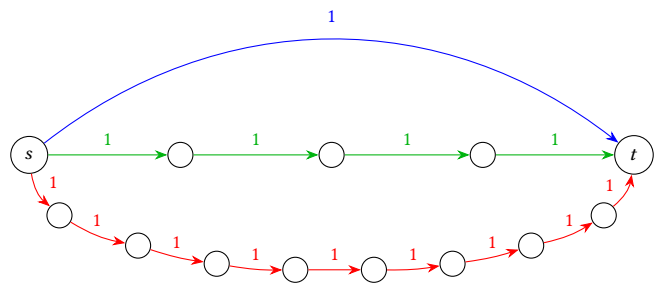



Fig. 1. Illustration for the given paths $p_1$ (blue), $p_2$ (green), $p_3$ (red) in Example 4. The edges are labeled with their respective linear cost coefficients.

*By Lemma 3, it suffices to show that there is no pure Nash equilibrium $x$ such that the induced path loads satisfy $\ell_{p_1}(x) \geq \ell_{p_2}(x) \geq \ell_{p_3}(x)$. That is, we need to analyze all* path load vectors $(\ell_{p_1}, \ell_{p_2}, \ell_{p_3}) \in \mathbb{N}^3$ *with $\ell_{p_1} + \ell_{p_2} + \ell_{p_3} = n = 5$ and $\ell_{p_1} \geq \ell_{p_2} \geq \ell_{p_3}$. Thus, it suffices to show that there is no PNE with path load vector among $(5, 0, 0), (4, 1, 0), (3, 2, 0), (3, 1, 1)$ and $(2, 2, 1)$. Table 1 provides an improving deviation for each of these candidate path load vectors, showing that the game has no PNE.*

Table 1. Improving deviations for the candidate path load vectors for the game in Example 4.

| *path loads* | *deviation (of a player using $p$ to $p'$, notation $p \to p'$)* | *resulting cost improvement* |
|---|---|---|
| $(5, 0, 0)$ | $p_1 \to p_2$ | $5a + B = 12 > 4 = 4 \cdot a$ |
| $(4, 1, 0)$ | $p_1 \to p_2$ | $4a + B = 11 > 8 = 4 \cdot 2a$ |
| $(3, 2, 0)$ | $p_1 \to p_3$ | $3a + B = 10 > 9 = 9 \cdot a$ |
| $(3, 1, 1)$ | $p_3 \to p_2$ | $9a = 9 > 8 = 4 \cdot 2a$ |
| $(2, 2, 1)$ | $p_2 \to p_1$ | $4 \cdot (2a + B/5) = 13,6 > 10 = 3a + B$ |

We would like to remark that in the above example, at least three paths are necessary to allow non-existence of pure Nash equilibria: If there are only two paths available for the players, it is an easy observation that any best response dynamic terminates with a pure Nash equilibrium. Furthermore, the paths cannot all have the same length: If the number of edges is the same for all paths, and all edge cost coefficients are identical, then we can in fact assume w.l.o.g. that all paths consist of a single edge only, and this then constitutes a (symmetric, singleton) matroid congestion game with identical cost coefficients for which we have shown existence of pure Nash equilibria in Theorem 2.

In the next example, we present a network congestion game with three parallel paths, each consisting of a single edge only (that is, a symmetric singleton matroid congestion game), but with *non-identical* cost coefficients. As in the previous example, the corresponding congestion game with an adversary does not admit a pure Nash equilibrium.

Example 5. *Consider a network consisting of three parallel edges $r_1, r_2$, and $r_3$. The edge cost coefficients are given by $a_{r_1} = 0$, $a_{r_2} = 2$, and $a_{r_3} = 5$. For an illustration see Figure 2. Assume we have $n = 5$ players with strategy set*

*$X_i = \{r_1, r_2, r_3\}$ for each player $i$, and the adversary's budget is $B = 6$. We proceed to show that the corresponding congestion game with an adversary has no pure Nash equilibrium.*

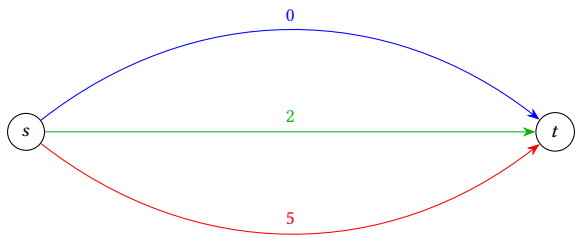



Fig. 2. Illustration for the resources $r_1$ (blue), $r_2$ (green), $r_3$ (red) in Example 5. The edges are labeled with their respective linear cost coefficients.

*Very similar as in the proof of Lemma 3, one can show that if there does not exist a pure Nash equilibrium $x$ such that $\ell_{r_1}(x) \geq \ell_{r_2}(x) \geq \ell_{r_3}(x)$, then the game does not admit a pure Nash equilibrium at all. Thus, it suffices to show that there is no PNE with* edge load profile $(\ell_{r_1}(x), \ell_{r_2}(x), \ell_{r_3}(x))$ *among* $(5, 0, 0)$, $(4, 1, 0)$, $(3, 2, 0)$, $(3, 1, 1)$ *and* $(2, 2, 1)$. *Table 2 provides an improving deviation for each of these candidate load profiles, showing that the game does not admit a PNE.*

Table 2. Improving deviations for the candidate profiles for the game in Example 5.

| load profile | deviation (of a player using $r$ to $r'$, notation $r \to r'$) | resulting cost improvement |
|---|---|---|
| $(5, 0, 0)$ | $r_1 \to r_2$ | $5a_{r_1} + B = 6 > 2 = a_{r_2}$ |
| $(4, 1, 0)$ | $r_1 \to r_2$ | $4a_{r_1} + B = 6 > 4 = 2a_{r_2}$ |
| $(3, 2, 0)$ | $r_1 \to r_3$ | $3a_{r_1} + B = 6 > 5 = a_{r_3}$ |
| $(3, 1, 1)$ | $r_3 \to r_2$ | $a_{r_3} = 5 > 4 = 2a_{r_2}$ |
| $(2, 2, 1)$ | $r_2 \to r_1$ | $2a_{r_2} + B/2 = 7 > 6 = 3a_{r_1} + B$ |

As we have seen in this section, pure Nash equilibria exist if the players' action spaces are bases of matroids and the resource cost coefficients are identical. However, if one of these two conditions is not fulfilled, existence cannot be guaranteed anymore. Motivated by this, we study *approximate* equilibria in the rest of this paper.

## 4 Computing $K$-approximate PNE

Motivated by the fact that PNE do not always exist, even for very simple forms of the underlying congestion game, we study in this section approximate PNE for the case of symmetric singleton congestion games, where each player is allowed to choose exactly one of the available resources. That is, $X_i = \{\{r\} : r \in R\}$ for all players $i \in N$ is assumed throughout this section. We show that $\alpha = K \approx 1.1974$ is the smallest possible $\alpha$ such that the existence

of an $\alpha$-PNE can be guaranteed for any such instance, where

$$K := \frac{1}{6}\left(1 + \sqrt[3]{109 - 6\sqrt{330}} + \sqrt[3]{109 + 6\sqrt{330}}\right) \tag{3}$$

is the unique solution of the equation $-x^3 + x^2/2 + 1 = 0$. To this end, we provide Algorithm 1 which efficiently computes a $K$-approximate PNE (see Theorem 10). This result is complemented by an instance where no $\alpha$-approximate PNE with $\alpha < K$ exists (see Theorem 14). As an easy consequence of the proof of Theorem 10, we also get that exact PNE are guaranteed to exist if $n \leq 4$ or $m \leq 2$ (see Corollary 12).

### 4.1 An Algorithm for Computing $\alpha$-Approximate Equilibria

For computing an $\alpha$-approximate PNE, we use the following basic approach. Starting with an empty game, we add the players one after another to the game, where a newly added player is always placed on a best response. If the addition of a new player makes some of the earlier added players "unhappy", meaning that they now have an $\alpha$-improving move, we let unhappy players deviate one after another to a best response until all players are happy again. Only then we add the next player and proceed as before.

Note that this approach has been used before to compute *exact* PNE, for example for player-specific costs or weighted congestion games on matroids [1, 47]. However, to the best of our knowledge, it has not been utilized in the context of approximate PNE. We believe that this technique will be useful for showing existence and computing approximate equilibria beyond the class of games that we analyze here.

We now formally describe our algorithm (see also Algorithm 1). We assume that the resource set $R = \{r_1, \ldots, r_m\}$ is ordered such that $a_1 \leq \cdots \leq a_m$ (where $a_j := a_{r_j}$). Since we need to consider strategy profiles for subsets of the players, let us extend the notion of a strategy profile to the set of vectors $x \in (R \cup \{0\})^n$, where $x_i = 0$ means that player $i$ has not yet been added to the game. Given such a vector $x \neq 0$, the cost $c_{x_i}(x)$ incurred to player $i$ with $x_i \neq 0$ is defined as the cost which is experienced by player $i$ in the game where only the players $j$ with $x_j \neq 0$ are present. Now define, for a strategy profile $x \in (R \cup \{0\})^n$, the following set of players $W_\alpha(x)$ who are "unhappy" with their strategy, meaning that they have an $\alpha$-improving move:

$$W_\alpha(x) := \{i \in N : x_i \neq 0 \text{ and } c_{x_i}(x) > \alpha \cdot c_r(r, x_{-i}) \text{ for a resource } r \in R\}.$$

In each iteration of the for-loop in line 2 of Algorithm 1, a new player $k$ is added to the game. We place $k$ on a best response (with respect to the strategies of the players $\{1, \ldots, k-1\}$ already added to the game). If there is more than one best response, we choose the one with smallest index (the *first* among the best responses if we view the resources ordered from $r_1$ to $r_m$), see lines 3 and 4. After having added player $k$, it may be the case that some players are not happy with their strategy, that is, $W_\alpha(x) \neq \emptyset$, where $x$ denotes the current strategy profile. In the while-loop starting in line 5, we iteratively choose a player who is "maximally unhappy", meaning that this player experiences maximum cost among all unhappy players, and let this player deviate to a best response, until all players in $\{1, \ldots, k\}$ are happy. If there is more than one resource with maximum cost among the resources used by unhappy players, we choose a player on the maximum cost resource with largest index (the *last* maximum cost resource w.r.t. the ordering from $r_1$ to $r_m$), and if there is more than one best response, we choose the one with smallest index (the *first* best response), see lines 7, 8, and 9. After this, we return to line 2 where the next player $k+1$ is added to the game.

It is clear that if Algorithm 1 terminates, the computed strategy profile is an $\alpha$-PNE. In the next subsection, we show that Algorithm 1 terminates for $\alpha = K$. Note that the special choices made by the algorithm in case of non-unique best responses or non-unique most expensive resources, together with the fact that resources are ordered such that $a_1 \leq a_2 \leq \cdots \leq a_m$, ensure that the loads are always decreasing along the resources, that is, $\ell_{r_1}(x) \geq \ell_{r_2}(x) \geq \cdots \geq \ell_{r_m}(x)$ always holds during the algorithm.

**Algorithm 1:** Computation of an $\alpha$-approximate PNE.

**Input:** Player set $N = [n]$, resource set $R = \{r_1, \ldots, r_m\}$, resource cost coefficients $0 \leq a_1 \leq \cdots \leq a_m$, budget $B > 0$, and $\alpha \geq 1$.

**Output:** $\alpha$-approximate pure Nash equilibrium $x$.

$x \leftarrow (0, 0, \ldots, 0)$;
**for** $k = 1, 2, \ldots, n$ **do**
  $t' \leftarrow \min\{t \in [m] : c_{r_t}(r_t, x_{-k}) \leq c_r(r, x_{-k}) \ \forall r \in R\}$;
  $x_k \leftarrow r_{t'}$;
  **while** $W_\alpha(x) \neq \emptyset$ **do**
    $R_\alpha(x) \leftarrow \{r \in R : x_i = r \text{ for some } i \in W_\alpha(x)\}$;
    $t^- \leftarrow \max\{t \in [m] : r_t \in R_\alpha(x) \text{ and } c_{r_t}(x) \geq c_r(x) \ \forall r \in R_\alpha(x)\}$;
    $i \leftarrow$ some player with $x_i = r_{t^-}$;
    $t^+ \leftarrow \min\{t \in [m] : c_{r_t}(r_t, x_{-i}) \leq c_r(r, x_{-i}) \ \forall r \in R\}$;
    $x_i \leftarrow r_{t^+}$;

We conclude this subsection with an illustrating example for Algorithm 1.

Example 6. *Consider the game with $n = 7$ players, $m = 5$ resources with cost coefficients $a_1 = 1$, $a_2 = a_3 = 4$, $a_4 = a_5 = 10$, and budget $B = 9$. Let $\alpha = K \approx 1.1974$ as specified in (3). We now describe the steps performed by Algorithm 1, see also Figure 3. Note that in the first six iterations of the for-loop, the players 1 to 6 are placed on the resources $r_1, r_2, r_3, r_1, r_4, r_5$, and no player changes during the while-loops. Only after player 7 is added (to resource $r_1$; see Figure 3 (a)), the players 5 and 6 become unhappy, because the cost for deviating to $r_2$ (or $r_3$) decreased from 12.5 to 8. Since the players 5 and 6 currently experience the same cost, but player 6's resource has a larger index, player 6 gets to change due to the tie-breaking rule of the algorithm. Both $r_2$ and $r_3$ are best responses, where $r_2$ has a smaller index, thus player 6 changes to $r_2$; see Figure 3 (b). The deviation of player 6 causes the players on $r_1$ to be unhappy since they can now improve their cost from 12 to 10 (more than a factor $K$) by deviating to $r_5$. Furthermore, player 5 can improve her cost from 10 to 8 (which is also more than a factor $K$) by deviating to $r_3$. By the tie-breaking rule of the algorithm, a player on $r_1$ gets to change since $12 > 10$. If one of the players on $r_1$, say player 7, deviates to $r_5$, the resulting strategy profile (see Figure 3 (c)) is a $K$-PNE since no player has a $K$-improving deviation anymore. (In fact, the profile is a $12.5/12 \approx 1.0417$-PNE.)*

*Note that if some player now deviates from $r_2$ to $r_1$ (which is a best response move), the resulting profile is not a $K$-PNE anymore—the situation is essentially as in Figure 3 (a). This shows that, starting from a $K$-PNE, a sequence of best responses does not necessarily yield a $K$-PNE.*

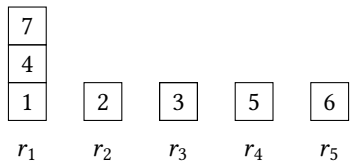
## 4.2 Termination of Algorithm 1 for $\alpha = K$

In this subsection, we prove that Algorithm 1 terminates for $\alpha = K$, and thus computes a $K$-approximate PNE. Furthermore, we show that the running time of Algorithm 1 can be bounded by $\min\{O(n^2m), O(nm^2)\}$.

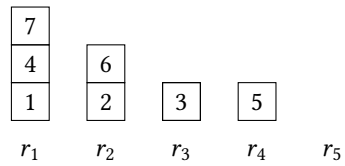
We start by stating some observations. While being simple, they turn out to be quite helpful in the subsequent analysis.

Observation 7. *During the course of Algorithm 1, the following statements always hold.*

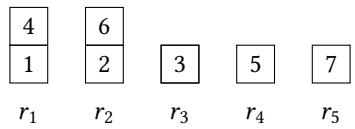
*(1) It is never beneficial to deviate to a resource where the load is smaller by one than the currently experienced load.*




*(a) Situation after player 7 is added. In the while-loop, player 6 wants to deviate to $r_2$ since $10 > K \cdot 8 \approx 9.58$.*




*(b) Situation after player 6 deviated to $r_2$. Player 7 now wants to deviate to $r_5$ since $12 > K \cdot 10 \approx 11.97$.*




*(c) Situation after player 7 deviated to $r_5$. The while-loop terminates since no player has a $K$-improving deviation anymore.*

Fig. 3. Example for Algorithm 1 with $n = 7$ players and five resources $r_1, \dots, r_5$. Player $i \in [n]$ is represented by a square containing number $i$.

*(2) If there are several resources with the same load, it is cheapest to deviate to the first such one (w.r.t. the ordering of the edges from $r_1$ to $r_m$).*
*(3) In the profile resulting from the addition of player $k$ to the game, any player which experiences the same load as player $k$ is happy.*
*(4) In the profile resulting from the deviation of some player $i$ during the while-loop, any player which experiences the same load as player $i$ is happy.*

*The statements (1) and (2) are direct consequences of the facts that loads are always decreasing along the resources while $a_1 \le a_2 \le \cdots \le a_m$. For (3) and (4), note first that if $x$ is a strategy profile and player $i$ unilaterally deviates to $r \neq x_i$, the* deviation cost *of player $i$ to $r$ (w.r.t. $x$) is $c_r(r, x_{-i}) = a_r(\ell_r(x) + 1) + \kappa^*_r(r, x_{-i})$ with*

$$\kappa^*_r(r, x_{-i}) = \begin{cases} B, & \text{if } \ell_r(x) = M, \\ 0, & \text{if } \ell_r(x) \le M - 3, \\ \frac{B}{p}, & \text{if } \ell_r(x) = M - 1 \text{ and } \ell_{x_i}(x) = M, \\ \frac{B}{p+1}, & \text{if } \ell_r(x) = M - 1 \text{ and } \ell_{x_i}(x) \le M - 1, \\ 0, & \text{if } \ell_r(x) = M - 2 \text{ and } M^{-1}(x) \neq \{x_i\}, \\ \frac{B}{q+2}, & \text{if } \ell_r(x) = M - 2 \text{ and } M^{-1}(x) = \{x_i\}, \end{cases} \tag{4}$$

*where $M := M(x)$ is the maximum load in $x$, $p := |M^{-1}(x)|$ is the number of maximum load resources, and $q := |\{r \in R : \ell_r(x) = M - 1\}|$ is the number of resources with load $M - 1$. From this we directly observe for two players $i$ and $j$ with $x_i \neq x_j$ but $\ell_{x_i}(x) = \ell_{x_j}(x)$, that the deviation costs to $r \notin \{x_i, x_j\}$ are the same. Now consider the profile $x$ resulting from the addition of player $k$ to the game (in Algorithm 1). Obviously, player $k$ (as well as any player using $x_k$) is happy and $x_k$ is the last resource with load $\ell_{x_k}(x)$ (w.r.t. the ordering from $r_1$ to $r_m$). Consequently, if there is a player $j$ with $x_j \neq x_k$ but $\ell_{x_j}(x) = \ell_{x_k}(x)$, the cost experienced on $x_j$ is not larger than the cost on $x_k$.*

*Thus player $j$ does not want to deviate to $x_k$, but also does not want to deviate to $r \neq x_k$ since player $k$ does not want to deviate to $r$. That is, player $j$ is also happy. Statement (4) follows by the same argumentation.*

We now turn to showing that Algorithm 1 terminates for $\alpha = K$. We need to prove that the while-loop terminates in each iteration $k \in \{1, \ldots, n\}$ of the for-loop. For $k = 1$, we only have one player, and this player is placed on the best response $r_1$. Therefore, $W_K(x) = W_K(r_1, 0, \ldots, 0) = \emptyset$ and the while-loop terminates without changing anything. For $k = 2$, the second player is placed on $r_1$ if $2a_1 + B \leq a_2 + B/2$, and is placed on $r_2$ otherwise. It is easy to see that in both cases, both players are happy and the while-loop immediately terminates. But for iteration $k \geq 3$, it may be the case that some players change their strategy during the while-loop. We show inductively that the while-loop also terminates in iteration $k \geq 3$. Note that since the while-loop terminated in iteration $k-1$, the players in $\{1, \ldots, k-1\}$ were happy before the next player $k$ was added to the game. That is, with respect to the profile $x$ directly before player $k$ is added, no player $i \in \{1, \ldots, k-1\}$ has a $K$-improving deviation, that is, an alternative resource $r$ with $c_{x_i}(x) > K \cdot c_r(r, x_{-i})$. But it may be the case that some players are unhappy *after* the addition of player $k$ and want to deviate. In the next lemma, we derive necessary properties for this to happen. The derived properties primarily depend on the loads, that is, the load of the resource where the new player $k$ is placed, as well as the loads of an unhappy player $i$'s strategy and of a corresponding $K$-improving deviation (that player $i$ wants to deviate to). For example, the first line in Table 3 means that if player $k$ is placed on a resource having load $M-2$, where $M$ denotes the maximum load before player $k$ was added, then player $i$ (who becomes unhappy) currently uses the unique maximum load resource and wants to deviate to a resource having load $M-2$. Similarly, the second line means that if player $k$ is placed on a resource having load $M-1$, then player $i$ wants to deviate from a resource having load at most $M-1$ to a resource having load $M-1$, and so on.

Lemma 8. *Let $x$ and $x' = (x'_k, x_{-k})$ denote the strategy profiles directly before and after a new player $k$ is added. Furthermore, let $M := M(x) = \max\{\ell_r(x) : r \in R\}$ denote the maximum load with respect to $x$, that is, before player $k$ is added.*

*Let $i \in \{1, \ldots, k-1\}$ be a player who becomes unhappy due to the addition of the new player $k$, that is, player $i$ has a $K$-improving deviation $r$ w.r.t. the profile $x'$. Then, regarding the loads of the resources $x'_k$, $x'_i = x_i$, and $r$, one of the cases shown in Table 3 (each row corresponds to one possible case) needs to hold.*

Table 3. Cases for the loads of the resources $x'_k, x_i$, and $r$, where $r$ is a $K$-improving deviation for player $i$ who becomes unhappy due to the addition of a new player $k$ on $x'_k$.

| $\ell_{x'_k}(x)$ | $\ell_{x_i}(x') = \ell_{x_i}(x)$ | $\ell_r(x') = \ell_r(x)$ | further conditions |
|---|---|---|---|
| $M-2$ | $M$ | $M-2$ | $M^{-1}(x) = \{x_i\}$ $(= M^{-1}(x'))$ |
| $M-1$ | $\leq M-1$ | $M-1$ | |
| $M$ | $\leq M$ | $M$ | |
| $M$ | $\leq M-1$ | $M-1$ | |

*Proof.* Let $i \in \{1, \ldots, k-1\}$ be a player who is unhappy with her strategy $x'_i = x_i$ in $x'$, and $r$ a corresponding $K$-improving deviation. Note that the players using $x'_k$ are happy w.r.t. $x'$ and thus $x_i \neq x'_k$ holds. Since $x'_k$ is the only resource where the load has changed compared to profile $x$, this implies $\ell_{x_i}(x) = \ell_{x_i}(x')$. From that we

easily get that $c_{x_i}(x') \leq c_{x_i}(x)$, that is, the addition of player $k$ to the game did not increase the cost experienced by player $i$ (the congestion part of the cost remains unchanged and the adversary part can only decrease). But since player $i$ was happy before player $k$ was added and in particular did not want to deviate to $r$ then, we thus get that the *deviation cost* (if player $i$ deviates to $r$) decreased, that is, $c_r(r, x'_{-i}) < c_r(r, x_{-i})$ holds. Clearly, the congestion part of the deviation cost did not decrease, since $\ell_r(x') \geq \ell_r(x)$. Therefore, the adversary part of the deviation cost needs to be smaller in $x'$ than it was in $x$, and this is only possible if it was positive in $x$. We thus get (compare (4)) that $\ell_r(x) \geq M - 2$, where $\ell_r(x) = M - 2$ is only possible if $x_i$ is the only resource with load $M$ in $x$. Furthermore, the adversary part of the deviation cost w.r.t. $x'$ needs to be smaller than $B$, which implies that $r$ cannot be a maximum load resource w.r.t. $x'$. From that we can easily deduce that $r \neq x'_k$: Otherwise, we get by the above that $\ell_{x'_k}(x) \geq M - 2$, but also $\ell_{x'_k}(x) \notin \{M-1, M\}$ (else $x'_k$ would be a maximum load resource w.r.t. $x'$). Thus the only possible case is $\ell_{x'_k}(x) = M - 2$, where, additionally, $x_i$ is the only resource with load $M$ in $x$. Then, however, the adversary part of the deviation cost to $r = x'_k$ w.r.t. $x'$ is $B$; contradiction. Therefore we have $r \neq x'_k$ and, consequently, $\ell_r(x) = \ell_r(x')$. We now distinguish between the three possible values for $\ell_r(x) \in \{M-2, M-1, M\}$ and derive the conditions stated in Table 3.

If $\ell_r(x) = M$, then $M(x') = M + 1$ and $\ell_{x'_k}(x) = M$ (since $r$ cannot be a maximum load resource w.r.t. $x'$).

If $\ell_r(x) = M - 1$, we get that $\ell_{x_i}(x) \leq M - 1$ since it is never beneficial to deviate to a resource where the load is smaller by one (compare Observation 7). Furthermore, $\ell_{x'_k}(x) \in \{M-1, M\}$ holds, otherwise the adversary part of the deviation cost to $r$ does not decrease.

It remains to analyze the case that $\ell_r(x) = M - 2$ and $x_i$ is the only resource with load $M$ in $x$. Since $x'_k \neq x_i$ we get that $\ell_{x'_k}(x) \neq M$. Furthermore, $\ell_{x'_k}(x) \geq M - 2$, otherwise the adversary part of the deviation cost to $r$ does not decrease. Finally, since player $i$ is unhappy whereas players experiencing the same load as player $k$, that is, with load $\ell_{x'_k}(x')$, are happy in $x'$ (see again Observation 7), one gets that $M = \ell_{x_i}(x') \neq \ell_{x'_k}(x') = \ell_{x'_k}(x) + 1$, and, altogether, $\ell_{x'_k}(x) = M - 2$. Altogether, one of the cases stated in Table 3 needs to hold. □

We have now seen that under the conditions stated in Table 3, some players might be unhappy after a new player $k$ is added to the game, and want to deviate in the while-loop. However, a deviation of one of these players during the while-loop may cause further players to be unhappy. For the termination of the while-loop, we need to show that after finitely many deviations, all players in $\{1, \ldots, k\}$ are happy again. We thus need to keep track of the set of unhappy players, as well as their corresponding $K$-improving deviations, during the course of the while-loop. To this end, we derive in the next lemma necessary properties for a player $j$ who gets a new $K$-improving deviation $r$ due to a deviation of some other player $i$ in the while-loop, where *new* means that $r$ was not $K$-improving for player $j$ before player $i$ deviated. Note that for this to happen, either player $j$'s own cost needs to increase, or the deviation cost of player $j$ to $r$ needs to decrease. In the proof, we analyze both cases, and derive the conditions shown in Table 4.

Lemma 9. *Assume that player $i \in \{1, \ldots, k\}$ deviates in some iteration of the while-loop of Algorithm 1, and denote the strategy profiles before and after this deviation by $x$ and $x' := (x'_i, x_{-i})$, respectively. Furthermore, let $M := M(x) = \max\{\ell_r(x) : r \in R\}$ denote the maximum load with respect to $x$, that is, before player $i$ deviated.*

*If player $j \in \{1, \ldots, k\} \setminus \{i\}$ gets a new $K$-improving deviation $r \neq x'_j = x_j$ due to player $i$'s deviation, then $x_j \neq x'_i$, and regarding the loads of the resources $x_i, x'_i, x_j$, and $r$, one of the cases shown in Table 4 (each row corresponds to one possible case) needs to hold.*

*Proof.* Clearly, $x'_i \neq x'_j = x_j$ since player $i$ is happy in $x'$ while player $j$ is not, and $i$ is the only player who changed strategy. Assume first that $c_{x_j}(x) < c_{x_j}(x')$, that is, player $j$'s own cost increased due to player $i$'s deviation. Since $x_j \neq x'_i$ implies $\ell_{x_j}(x) \geq \ell_{x_j}(x')$, the congestion part of player $j$'s cost did not increase, thus the adversary part needed to increase. In particular, the adversary part w.r.t. $x'$ cannot be zero, which implies that $x_j$

Table 4. Cases for the loads of the resources $x_i, x_i', x_j$, and $r$, where $r$ is a new $K$-improving deviation for player $j$ after player $i$ deviated from $x_i$ to $x_i'$. In the last row, $\ell_r(x') \leq M-3$ is only possible if $r = x_i$.

| $\ell_{x_i}(x)$ | $\ell_{x_i'}(x)$ | $\ell_{x_j'}(x') = \ell_{x_j}(x')$ | $\ell_r(x')$ | further conditions |
|---|---|---|---|---|
| $M$ | $M$ | $\leq M$ | $M$ | |
| | | $\leq M-1$ | $M-1$ | |
| | $\leq M-2$ | $M$ | $M$ or $\leq M-2$ | $M^{-1}(x) \supsetneq \{x_i\}$ |
| | $\leq M-3$ | $M-1$ | $M-1$ or $\leq M-3$ | $M^{-1}(x) = \{x_i\}$ |
| $\leq M-1$ | $M$ | $\leq M$ | $M$ | |
| | | $\leq M-1$ | $M-1$ | |
| | $M-1$ | $\leq M-1$ | $M-1$ | |
| $\leq M-2$ | $M-2$ | $M$ | $M-2$ or $\leq M-3$ ($r = x_i$) | $M^{-1}(x) = \{x_j\}$ |

has maximum load in $x'$, and the adversary part w.r.t. $x$ cannot be $B$. We distinguish between the different values that the maximum load $M(x')$ can take. Since $M(x') = \ell_{x_j}(x') \leq \ell_{x_j}(x) \leq M$, we have that $M(x') \in \{M-1, M\}$.

If $M(x') = M-1$, then player $i$ moved from the only resource with load $M$ to a resource with load at most $M-2$. Even more, we can exclude the case that $\ell_{x_i'}(x) = M-2$: If player $i$ moves to a resource with load $M-2$, then all players having load $M-1$ are happy in $x'$ (since they have the same load as player $i$, see Observation 7), but we know that player $j$ is unhappy and has load $M(x') = M-1$ in $x'$; contradiction.[1] Furthermore, $x_j \neq x_i$ and thus $\ell_{x_j}(x) = \ell_{x_j}(x') = M-1$ since the adversary part of player $j$'s cost in $x$ cannot be $B$. Finally, $\ell_r(x') \leq M-1 = M(x')$ and $\ell_r(x') \neq M-2$ since it is never beneficial to move to a resource where the load is smaller by one and $\ell_{x_j}(x') = M-1$. Consequently, either $\ell_r(x') = M-1$ or $\ell_r(x') \leq M-3$. We have thus derived the conditions stated in the fourth row of Table 4.

If $M(x') = M$, we get that $M \geq \ell_{x_j}(x) \geq \ell_{x_j}(x') = M$, that is, $\ell_{x_j}(x) = \ell_{x_j}(x') = M$ (and, consequently, $x_j \neq x_i$). Since the adversary part of player $j$'s cost needs to increase, we have that the number of resources with load $M$ needs to decrease. But this means that player $i$ moved from a resource with load $M$ to a resource with load at most $M-2$. In particular, there are at least two resources with load $M$ in $x$ (namely $x_i$ and $x_j$). Noting that $\ell_r(x') \leq M(x') = M$, and that player $j$ does not want to deviate to a resource where the load is smaller by one, that is, where the load is $M-1$, we obtain the conditions in the third row of Table 4.

Now consider the case that player $j$'s own cost did not increase due to player $i$'s deviation. Then, the deviation cost of player $j$ to $r$ is decreased. Consider first the case that $r \neq x_i$. In that case $\ell_r(x) \leq \ell_r(x')$, and consequently, the congestion part of the deviation cost did not decrease. Therefore, the adversary part is decreased. In particular, the adversary part was positive w.r.t. $x$, and it is smaller than $B$ w.r.t. $x'$. This shows that $\ell_r(x) \geq M-2$ and $\ell_r(x') < M(x')$. We distinguish between the different values for $M(x') \in \{M-1, M, M+1\}$.

[1]When we speak of a player having a certain load, we mean that the player's current strategy has that load.

If $M(x') = M + 1$, we have that $\ell_{x'_i}(x) = M$ and $M - 2 \leq \ell_r(x) \leq \ell_r(x') \leq M$. Furthermore, $\ell_r(x') = M - 2$ is only possible if $r \neq x'_i$ and $x_j$ was the only resource with load $M$ in $x$ (otherwise the adversary part of deviating to $r$ is zero in $x$). But this shows $x_j = x'_i$; contradiction. Thus $\ell_r(x') \in \{M - 1, M\}$. We are thus in one of the cases corresponding to rows 1, 2, 5 and 6 of Table 4 (we again use that it is never beneficial to deviate to a resource where the load is smaller by one).

If $M(x') = M$, we get $\ell_{x'_i}(x) \leq M - 1$ and $M - 2 \leq \ell_r(x) \leq \ell_r(x') \leq M - 1$. We distinguish between the two different values that $\ell_r(x) \in \{M - 2, M - 1\}$ can take. If $\ell_r(x) = M - 2$, then $x_j$ is the only resource with load $M$ in $x$. Since the maximum load did not decrease, we get from this that $x_i \neq x_j$ and consequently, $\ell_{x_i}(x) \leq M - 1$ and $\ell_{x_j}(x') = \ell_{x_j}(x) = M$. Since it is not beneficial to deviate to a resource with load smaller by one, we get $\ell_r(x') = M - 2$. This shows $\ell_{x'_i}(x) \in \{M - 1, M - 2\}$ (the number of resources with load $M$, or the number of resources with load $M - 1$, needs to be larger than before). However, the case $\ell_{x'_i}(x) = M - 1$ can be excluded (otherwise, players with load $M$ in $x'$ would be happy, but player $j$ is not happy). Consequently, $\ell_{x'_i}(x) = M - 2$ and $\ell_{x_i}(x) \leq M - 2$. We have thus derived a situation which is captured by the last row in Table 4. Now consider the case that $\ell_r(x) = M - 1$, and, consequently, $\ell_r(x') = M - 1$ while $\ell_{x_j}(x') \leq M - 1$. Recall that $\ell_{x'_i}(x) \leq M - 1$ and if $\ell_{x'_i}(x) = M - 1$, we know that $\ell_{x_i}(x) \leq M - 1$ and we are thus in the case described by the seventh row of Table 4. On the other hand, $\ell_{x'_i}(x) \leq M - 2$ is not possible: In that case, the number of maximum load resources can only be smaller in $x'$ than in $x$. However this contradicts the fact that the deviation cost of player $j$ to $r$ decreased: We know that $\ell_{x_j}(x') \leq M - 1$. Now if $\ell_{x_j}(x) \leq M - 1$, the deviation cost to $r$ can only be larger. Else, we have that $\ell_{x_j}(x) = M$ thus $x_i = x_j$. But then, the deviation cost to $r$ is unchanged.

If $M(x') = M - 1$, player $i$ moved from the only resource with load $M$ in $x$ to a resource with load at most $M - 2$. Furthermore, $M - 2 \leq \ell_r(x) \leq \ell_r(x') < M(x')$ implies $\ell_r(x') = \ell_r(x) = M - 2$. But since the adversary part of the deviation cost to $r$ w.r.t. $x$ needs to be positive, this is only possible if $x_j$ is the only resource with load $M$ in $x$, that is, $x_j = x_i$, and consequently, $\ell_{x_j}(x') = M - 1$. However, it is never beneficial to deviate to a resource with load smaller by one; contradiction.

It remains to analyze the case that $r = x_i$. In particular, $x_i \neq x_j$ and $\ell_{x_j}(x) = \ell_{x_j}(x')$. Since we assumed that player $j$'s own cost did not increase, we have $c_{x_j}(x) \geq c_{x_j}(x')$. If, additionally, $c_{x_i}(x_i, x'_{-j}) \geq c_{x_i}(x)$ and $c_{x'_i}(x'_i, x_{-i}) \geq c_{x'_i}(x'_i, x_{-j})$ hold, we can derive a contradiction as follows. Using that player $i$ deviated from $x_i$ to $x'_i$ and player $j$ wants to deviate from $x_j$ to $x_i$, we get

$$c_{x_j}(x) \geq c_{x_j}(x') > Kc_{x_i}(x_i, x'_{-j}) \geq Kc_{x_i}(x) > K^2 c_{x'_i}(x'_i, x_{-i}) > Kc_{x'_i}(x'_i, x_{-i}) \geq Kc_{x'_i}(x'_i, x_{-j}).$$

This means that $x'_i$ is a $K$-improving deviation for player $j$ w.r.t. $x$, too, and furthermore, $c_{x_j}(x) > c_{x_i}(x)$. But this contradicts the fact that player $i$ is an unhappy player with largest cost in $x$ (by the rules of Algorithm 1). Thus it is not possible that both $c_{x_i}(x_i, x'_{-j}) \geq c_{x_i}(x)$ and $c_{x'_i}(x'_i, x_{-i}) \geq c_{x'_i}(x'_i, x_{-j})$ hold. Consider first the case that $c_{x_i}(x_i, x'_{-j}) < c_{x_i}(x)$. Clearly, the congestion parts of these two costs are the same since the load on $x_i$ is the same, thus the adversary part of $c_{x_i}(x_i, x'_{-j})$ needs to be smaller. This implies that the adversary part of $c_{x_i}(x)$ is positive. In particular, $\ell_{x_i}(x) = M$, which implies $\ell_{x'_i}(x) \neq M - 1$ and $\ell_{r=x_i}(x') = M - 1$. From the last, we also get $\ell_{x_j}(x') \leq M - 1$. Furthermore, since the adversary part of $c_{x_i}(x_i, x'_{-j})$ needs to be smaller, either the maximum load in $(x_i, x'_{-j})$ is larger than $M$ (that is, $M + 1$, and $\ell_{x'_i}(x) = M$), or the number of resources with load $M$ increased (but this is only possible if $\ell_{x'_i}(x) = M - 1$; contradiction). We have thus derived the conditions stated in the second row of Table 4. Now assume that $c_{x'_i}(x'_i, x_{-i}) < c_{x'_i}(x'_i, x_{-j})$, that is, the adversary part of the deviation cost to $x'_i$ is smaller for player $i$ than for player $j$. This means $\ell_{x'_i}(x) \in \{M - 1, M - 2\}$ (compare (4)). If $\ell_{x'_i}(x) = M - 1$, we get that $\ell_{x_j}(x) = M$ and $\ell_{x_i}(x) \leq M - 1$. But then, all players with load $M$ are happy in $x'$ (since $\ell_{x'_i}(x') = M$), contradicting the fact that player $j$ is unhappy. If $\ell_{x'_i}(x) = M - 2$, we get that $x_j$ is the only resource with load $M$ in $x$ and $\ell_{x_i}(x) \neq M - 1$. Consequently, $\ell_{x_i}(x) \leq M - 2$ and $\ell_{r=x_i}(x') \leq M - 3$. We thus have a situation which is captured by the last row of Table 4. □

We are now ready to prove the main result of this section.

Theorem 10. *For $\alpha = K$, where*

$$K = \frac{1}{6}\left(1 + \sqrt[3]{109 - 6\sqrt{330}} + \sqrt[3]{109 + 6\sqrt{330}}\right) \approx 1.1974$$

*is the unique solution of the equation $-x^3 + x^2/2 + 1 = 0$, Algorithm 1 computes a $K$-PNE.*

*Proof.* It suffices to show that in iteration $k \geq 3$ of the for-loop, the while-loop terminates. Let $x$ and $x'$ denote the profiles directly before and after the new player $k$ is added. If all players are happy with their strategy in $x'$, the statement follows; thus assume that at least one player becomes unhappy due to the addition of player $k$. Let $M := M(x) = \max\{\ell_r(x) : r \in R\}$ be the maximum load in $x$, that is, before player $k$ is added, and recall that $x'_k$ denotes the resource to which player $k$ is added. Consequently, $\ell_{x'_k}(x)$ denotes the load of $x'_k$ before player $k$ was added. By Lemma 8, we have $\ell_{x'_k}(x) \in \{M-2, M-1, M\}$, and we also know the following about unhappy players and their $K$-improving deviations: If $\ell_{x'_k}(x) = M-2$, unhappy players use the only resource with load $M$ and want to move to resources with load $M-2$. If $\ell_{x'_k}(x) = M-1$, unhappy players want to move to resources with load $M-1$ (and currently experience load $\leq M-1$ since it is never beneficial to move to a resource with load smaller by one). Finally, if $\ell_{x'_k}(x) = M$, unhappy players use resources with load $\leq M$ and want to deviate to resources with load $M-1$ or load $M$ (if they want to move to load $M-1$, their current load is $\leq M-1$).

We now analyze each of these three cases, and make repeated use of Lemma 9 and Observation 7. Figure 4 illustrates the complete case distinction that we carry out in the proof.

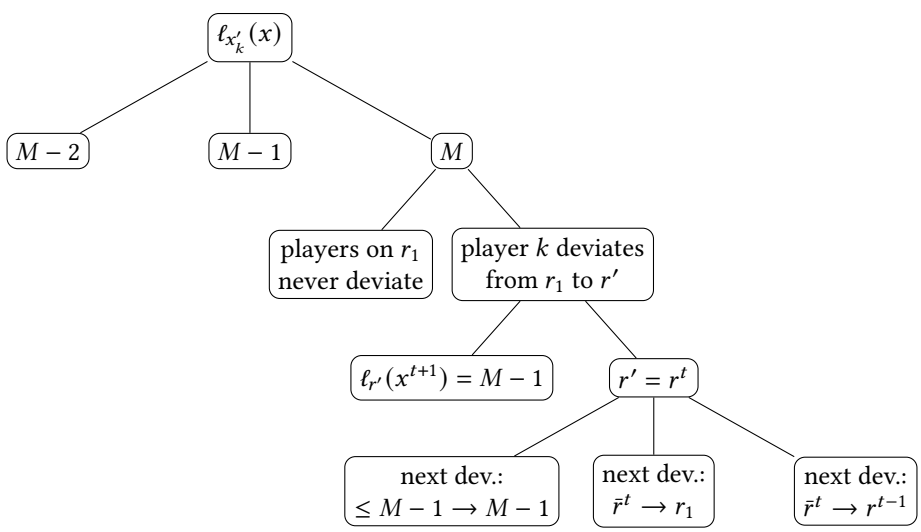



Fig. 4. Illustration for the case distinction in the proof of Theorem 10.

*Case $\ell_{x'_k}(x) = M-2$.* We know that unhappy players use the only resource with load $M$ and want to deviate to resources with load $M-2$. Assume that one of these players, say player $i$, deviates in the while-loop from $x'_i = x_i$ to $r$, and denote the profile resulting from player $i$'s deviation by $x'' := (r, x'_{-i})$. Figure 5 illustrates the situation. We now show that all players are happy in $x''$ and thus the while-loop terminates after one iteration.

Note first that all players having the same load in $x''$ as player $i$, that is, with load $M-1$, are happy in $x''$ (cf. Observation 7). In particular, all players who were unhappy before player $i$ deviated are now happy, since they used the only resource with load $M$ and the load on that resource decreased to $M-1$. Therefore, if there is an unhappy player $j$ with $K$-improving deviation $r'$ in $x''$, then player $j$ was happy before player $i$ deviated and $r'$ is a *new* $K$-improving deviation for player $j$ in $x''$. However, applying Lemma 9 for the deviation of player $i$ (thus $x, x'$ in the statement of the lemma are now $x', x''$) shows that there cannot be a new $K$-improving deviation: Note that $\ell_{x'_i}(x') = M$, $\ell_{x''_i}(x') = M-2$, and resource $x'_i$ is the only resource with load $M$ in $x'$, but this case does not occur in Table 4 and thus there are no new $K$-improving deviations. We conclude that all players are happy in $x''$ and the while-loop terminates after one iteration.

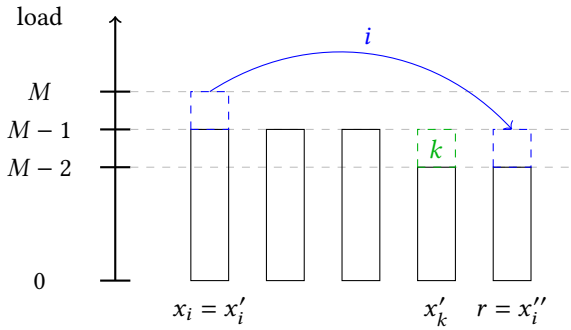



Fig. 5. Situation in Case $\ell_{x'_k}(x) = M-2$: Player $k$ is placed on the first resource with load $M-2$, and player $i$ deviates from the only resource with load $M$ to resource $r$ with load $M-2$.

*Case* $\ell_{x'_k}(x) = M-1$. We know that unhappy players want to deviate to resources with load $M-1$ (and currently experience load $\leq M-1$). Assume as above that one of these players, say player $i$, deviates in the while-loop from $x'_i = x_i$ to $r$, and denote the profile resulting from player $i$'s deviation by $x'' := (r, x'_{-i})$. Figure 6 illustrates the situation. If all players are happy in $x''$, the while-loop terminates after one iteration, thus we may assume that there is an unhappy player $j$ with $K$-improving deviation $r'$. We show that $\ell_{r'}(x'') = M-1$, that is, player $j$ wants to move to a resource with load $M-1$ (which implies that the current load of player $j$ is $\leq M-1$ since it is never beneficial to deviate to a resource with load smaller by one).

If $r'$ is a *new* $K$-improving deviation for player $j$, we get from Lemma 9 applied for the deviation of player $i$ (thus $x, x', r$ in the statement of the lemma are now $x', x'', r'$) that $\ell_{r'}(x'') = M-1$ (seventh row of Table 4). Now assume that $r'$ was already $K$-improving for player $j$ before player $i$ deviated. Consequently, the load on $r'$ was $M-1$ before player $i$ deviated. It remains to argue that the load on $r'$ did not change by player $i$'s deviation. Clearly, the only resources with different load are $x'_i$ and $x''_i = r$. Furthermore, $r' = x'_i$ is not possible (since player $i$ was chosen as the deviating player in the while-loop, player $i$ experiences as least as high cost in $x'$ than player $j$, $c_{x'_i}(x') \geq c_{x'_j}(x')$, and consequently, $x'_i$ is not $K$-improving for player $j$ in $x'$). Finally, since $r_1$ (the best option with load $M$) is not $K$-improving for player $j$ (that would be a new $K$-improving deviation but all those have load $M-1$, as we already showed), deviating to any resource with load $M$ is not $K$-improving. Thus $r' = x''_i$ is also not possible, showing altogether that players who are unhappy in $x''$ want to change to load $M-1$ (and use resources with load $\leq M-1$). However this is the same situation as before player $i$ deviated, thus by repeating the same argumentation we get that all future deviations move to resources with load $M-1$. But since each deviation decreases the number of resources with load $M-1$ at least by one and there are at most $m-2$ resources with load $M-1$ in $x'$, we conclude that the while-loop terminates after $O(m)$ iterations.

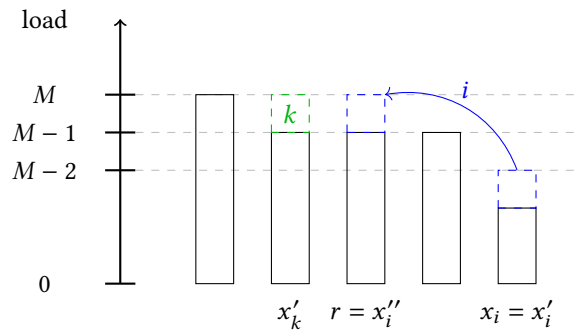
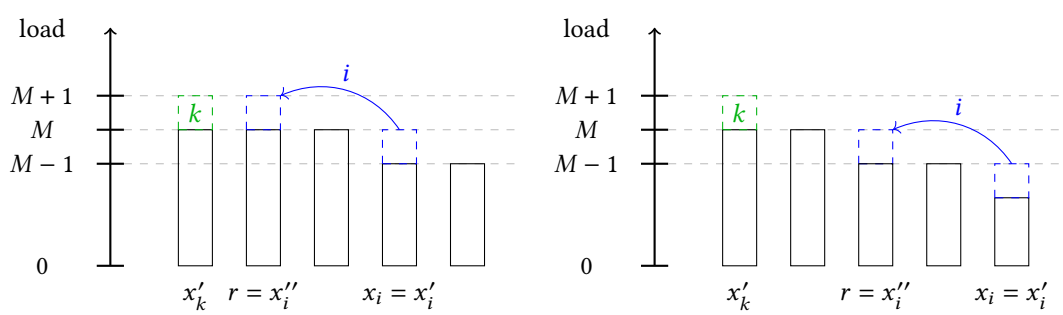



Fig. 6. Situation in Case $\ell_{x'_k}(x) = M - 1$: Player $k$ is placed on the first resource with load $M - 1$, and player $i$ deviates from a resource with load $\leq M - 1$ to resource $r$ with load $M - 1$. (There might be more than one resource with load $M$ before player $k$ was added.)

*Case* $\ell_{x'_k}(x) = M$. Note that in this case, the maximum load increased from $M$ to $M+1$, and in particular, $x'_k = r_1$ is the only resource with load $M + 1$ in $x'$.

We know that unhappy players use resources with load $\leq M$ and want to deviate to resources with load $M - 1$ or load $M$ (if they want to move to load $M - 1$, their current load is $\leq M - 1$). Assume as above that one of these players, say player $i$, deviates in the while-loop from $x'_i = x_i$ to $r$, and denote the profile resulting from player $i$'s deviation by $x'' := (r, x'_{-i})$. We distinguish between the two different values that the load $\ell_r(x') \in \{M - 1, M\}$ can take (see Figure 7 for illustrations of both cases).




(a) Player $i$ deviates from a resource with load $\leq M$ to resource $r$ with load $M$.

(b) Player $i$ deviates from a resource with load $\leq M - 1$ to resource $r$ with load $M - 1$.

Fig. 7. Situation in Case $\ell_{x'_k}(x) = M$: Player $k$ is placed on $r_1$.

First consider the case that $\ell_r(x') = M$ (Figure 7 (a)), and assume there is a player $j$ who is unhappy in $x''$ with $K$-improving deviation $r'$. We want to show that $\ell_{r'}(x'') \in \{M - 1, M\}$. If $r'$ is a new $K$-improving deviation for player $j$, we get by Lemma 9 applied for player $i$'s deviation (thus $x, x', M, r$ in the statement of the lemma are now $x', x'', M + 1, r'$), that $\ell_{r'}(x'') = M$ (see the seventh row in Table 4). Now consider the case that $r'$ was already $K$-improving for player $j$ in $x'$. Then $\ell_{r'}(x') \in \{M - 1, M\}$. If the load on $r'$ is unchanged, we get the desired property. Clearly, the only resources with changed load are $x'_i$ and $r = x''_i$. Moreover, $r' \neq x'_i$ (since

$c_{x'_i}(x') \geq c_{x'_j}(x')$ and, consequently, $x'_i$ is not $K$-improving for player $j$ in $x'$). Finally, since $r_1$ (the best option with load $M+1$) is not $K$-improving for player $j$ (that would be a new $K$-improving deviation but all those have load $M$, as we already showed), deviating to any resource with load $M+1$ is not $K$-improving. Thus $r' \neq x''_i$. Altogether, if $\ell_r(x') = M$, players who are unhappy in $x''$ use resources with load $\leq M$ (since all players with the same load as player $i$, that is, with load $M+1$, are happy) and want to deviate to resources with load $M-1$ or load $M$ (if they want to move to load $M-1$, their current load is $\leq M-1$).

For the case that $\ell_r(x') = M-1$ (see Figure 7 (b)), assume again there is a player $j$ who is unhappy in $x''$ with $K$-improving deviation $r'$. Note that $\ell_{x''_j}(x'') \neq M$ since all players with the same load as player $i$, that is, with load $M$, are happy in $x''$. If $r'$ is a new $K$-improving deviation for player $j$, we get from Lemma 9 that player $j$ uses $r_1$ and either wants to deviate to a resource with load $M-1$, or to $x'_i$ (see the last row in Table 4). If $r'$ was already $K$-improving for player $j$ in $x'$, we know as before that $\ell_{r'}(x') \in \{M-1, M\}$. Furthermore, the load on $r'$ cannot be smaller in $x''$ (since $r' \neq x'_i$ by the same argumentation as above), and it can only be larger if $r' = x''_i$, but in that case the load is now $M$. Thus we still have $\ell_{r'}(x'') \in \{M-1, M\}$. Therefore, if $\ell_r(x') = M-1$, players who are unhappy in $x''$ either use $r_1$ and want to deviate to a resource with load $M-1$ or $x'_i$, or they use resources with load $\leq M-1$ and want to deviate to resources with load $M-1$ or $M$.

Using the above, we can argue that if the players on $r_1$ never deviate during the while-loop, then the only changes are from resources having load $\leq M$ to resources with load $M$, or from $\leq M-1$ to $M-1$. But this terminates after $O(m)$ many deviations (each deviation, except one from load $\leq M-2$ to $M-1$, strictly decreases the number of resources with load in $\{M-1, M\}$; and there can also be at most $O(m)$ many deviations from load $\leq M-2$ to $M-1$).

Thus we can now assume that in some iteration of the while-loop, a player deviates from $r_1$ to a resource $r'$. Consider the first such change, that is, all changes before have been from load $\leq M$ to load $M$ or from $\leq M-1$ to $M-1$. More exactly, note that the changes before cannot include a change to load $M$, because after such a change, all players having load $M+1$ are happy and never become unhappy afterwards (see Lemma 9 and note that $r_1$ is not the only resource with load $M+1$ anymore). Therefore, all changes before were moves from load $\leq M-1$ to $M-1$ (and there was at least one such change). Let $i_1, \ldots, i_t$ be the corresponding sequence of deviating players, where $i_1 = i$ is the first, and $i_t$ the last player deviating before player $k$ deviates (we can assume w.l.o.g. that player $k$ is the player deviating from $r_1$ to $r'$). Let $r^s$ and $\bar{r}^s$ denote player $i_s$'s resources before and after changing, for $s = 1, \ldots, t$, and let $x^{s+1}$ be the strategy profile resulting from $i_s$'s change. Finally, let $x^{t+2} := (r', x^{t+1}_{-k})$ be the strategy profile resulting from $x^{t+1}$ due to player $k$'s deviation from $r_1$ to $r'$. Figure 8 shows an illustration for the situation before player $k$ deviates.

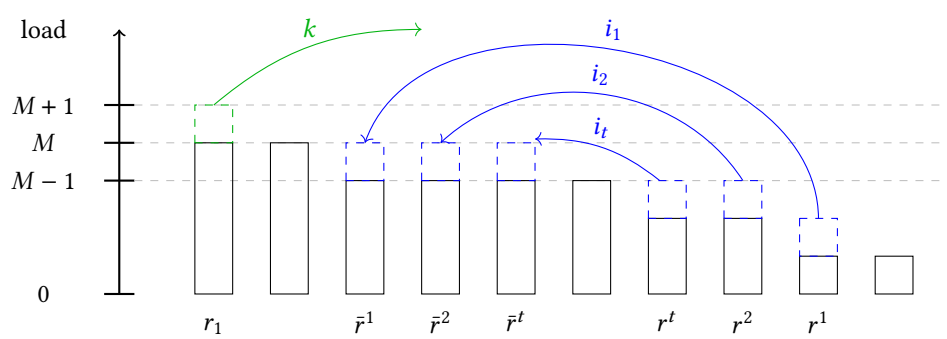



Fig. 8. Situation before player $k$ deviates (with $t = 3$).

Note that $r_1$ is the only resource with load $M+1$ in $x^{t+1}$. Furthermore, all players having load $M$ are happy in $x^{t+1}$ (since the last deviation was to load $M-1$), thus the only unhappy players in $x^{t+1}$ (except from the players using $r_1$) might want to change from load $\leq M-1$ to load $M-1$ or to load $M$. Now consider the situation after player $k$ deviated to $r'$. Recall that player $k$ either deviates to a resource with load $M-1$, or to $r^t$ (smallest resulting cost among all resources where the players $i_1, \ldots, i_t$ deviated from, and smallest index in case of non-uniqueness).

*Subcase* $\ell_{r'}(x^{t+1}) = M-1$. If player $k$ deviates to a resource with load $M-1$, then all players having load $M$ are happy in $x^{t+2}$. Thus consider a player $j$ with load $\leq M-1$. Comparing the current situation, that is, profile $x^{t+2}$, with the situation before player $k$ was added to the game, that is, profile $x$, we observe that player $j$'s private cost can only be smaller in $x^{t+2}$ than in $x$. Furthermore, the deviation cost to a resource $\tilde{r}$ can only be smaller in $x^{t+2}$ than in $x$ if the load of $\tilde{r}$ is $M-1$, or if $\tilde{r} \in \{r^1, r^2, \ldots, r^t\}$. Since player $j$ was happy in $x$, we get that if $\tilde{r}$ is a $K$-improving deviation for player $j$ in $x^{t+2}$, then either the load of $\tilde{r}$ is $M-1$, or $\tilde{r} \in \{r^1, r^2, \ldots, r^t\}$. However, the latter case can also be excluded, since player $j$ did not want to deviate to such a resource before player $k$ deviated ($K$-improving deviations had load $M-1$ or $M$), and player $k$'s deviation did neither influence player $j$'s private cost, nor the deviation cost to one of these resources. Consequently, unhappy players in $x^{t+2}$ have load $\leq M-1$ and want to deviate to a resource with load $M-1$. Using very similar arguments as before, one observes that in fact *all* future changes are from load $\leq M-1$ to load $M-1$, showing that the while-loop terminates after $O(m)$ iterations.

*Subcase* $r' = r^t$. It remains to analyze the case that player $k$ deviates from $r_1$ to $r^t$ (the resource where player $i_t$ deviated from). Assume player $j$ is unhappy after that change with $K$-improving deviation $\tilde{r}$. Clearly, the load of player $j$'s resource is $\leq M$, and if it is equal to $M$, the load of $\tilde{r}$ is either $M$ or $\leq M-2$. If, on the other hand, player $j$ uses a resource with load $\leq M-1$, then we can show by a very similar argumentation as in the last paragraph that the load of $\tilde{r}$ is $M-1$. Therefore, there are three possible cases regarding the next deviation in the while-loop: Either a player deviates from $\bar{r}^t$ (the resource where player $i_t$ deviated to, see Figure 8) to $r_1$, or from $\bar{r}^t$ to a resource with load $\leq M-2$, or from a resource with load $\leq M-1$ to load $M-1$ (note that $\bar{r}^t$ has largest cost and largest index, while $r_1$ has smallest cost and smallest index, among the resources with load $M$).

In the last case, all players with load $M$ are happy after the change, and by very similar arguments as before, the only further moves can be from load $\leq M-1$ to load $M-1$. This terminates after $O(m)$ iterations.

The case that a player moves from $\bar{r}^t$ to $r_1$ (see Figure 9 for an illustration) can be excluded since it leads to the following contradiction. Let $\ell_{r^t}$ denote the load of resource $r^t$ directly before player $i_t$ deviated from it. Furthermore note that there are at least two resources with load $M$ in $x^{t+2}$ ($r_1$ and $\bar{r}^1$). Then, the following three inequalities hold.

$$a_{r^t} \cdot \ell_{r^t} > K \cdot a_{\bar{r}^t} \cdot M \tag{5}$$

$$a_1 \cdot (M+1) + B > K \cdot a_{r^t} \cdot \ell_{r^t} \tag{6}$$

$$a_{\bar{r}^t} \cdot M + B/2 > K \cdot (a_1 \cdot (M+1) + B). \tag{7}$$

Using that (7) is equivalent to $Ka_{\bar{r}^t} \cdot M > K^2 a_1 \cdot (M+1) + (K^2 - K/2)B$ and combining this with (5) and (6) yields

$$K^2 a_1 \cdot (M+1) + (K^2 - K/2)B < a_{r^t} \cdot \ell_{r^t} < \tfrac{a_1 \cdot (M+1)+B}{K}.$$

From this we conclude the contradiction

$$0 \leq (K^3 - 1)a_1 \cdot (M+1) < (-K^3 + K^2/2 + 1)B = 0,$$

where we additionally used that $K^3 > 1$ and $-K^3 + K^2/2 + 1 = 0$ by definition of $K$.

Therefore, the only remaining case is that a player on $\bar{r}^t$ changes to a resource $r''$ with load $\leq M-2$. We will first show that $r'' = r^{t-1}$ holds. Note that the cost for deviating to $r''$ is strictly smaller than the cost for deviating to $r_1$ (otherwise, one would change to $r_1$). Therefore, $r''$ needs to have strictly smaller load now than it had in $x$

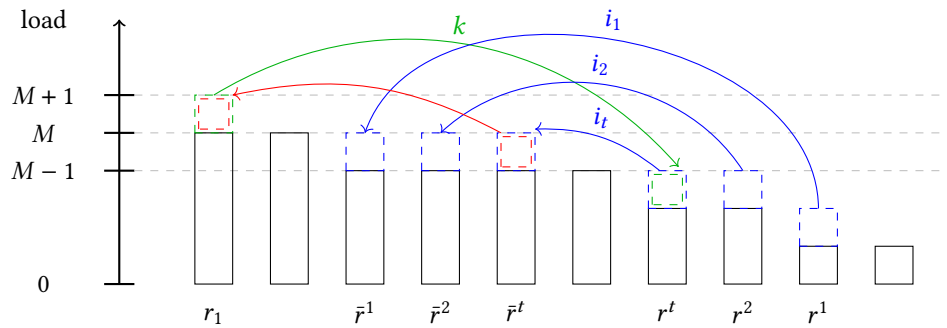



Fig. 9. Illustration for the case $\bar{r}^t \to r_1$ (with $t = 3$).

(before player $k$ was added to the game, otherwise, player $k$ would be placed on $r''$ rather than on $r_1$). This shows that $r'' \in \{r^1, \dots, r^{t-1}\}$ needs to hold, and since deviating to $r^{t-1}$ is cheapest and $r^{t-1}$ has the smallest index among the cheapest deviations, the assertion follows.

Consider the situation after the change from $\bar{r}^t$ to $r^{t-1}$ (of some player $j$, see Figure 10 for illustration). We will show that unhappy players have load $M$ (consequently, they want to deviate to a resource with load $M$ or load $\leq M-2$). Assume, by contradiction, that there is an unhappy player $h$ on a resource with load $\leq M-1$ (and $K$-improving deviation $\hat{r}$). Compare the current situation with the situation before player $k$ was added to the game. Since the cost of player $h$'s resource can only be smaller and the players on that resource were happy in $x$, we get that the deviation cost to $\hat{r}$ needs to be strictly smaller than it was before player $k$ was added to the game. That is, either the load of $\hat{r}$ is $M-1$, or $\hat{r} \in \{r^1, \dots, r^{t-2}\}$. However, the order in which the players moved during the while-loop shows that it is neither beneficial for player $h$ to move to $\bar{r}^t$ (the best option with load $M-1$) nor to $r^{t-2}$ (the best option in $\{r^1, \dots, r^{t-2}\}$); contradiction. Therefore, we get that unhappy players have load $M$ (and want to move to a resource with load $M$ or load $\leq M-2$). Consequently, the next move is from $\bar{r}^{t-1}$ to $r_1$, or from $\bar{r}^{t-1}$ to a resource with load $\leq M-2$. The former case leads to a contradiction (note that in this case, (5)–(7) hold since $a_{\bar{r}^t} \geq a_{\bar{r}^{t-1}}$). Thus we are in the situation that a player on $\bar{r}^{t-1}$ changes to a resource with load $\leq M-2$. As above, we can show that the deviation is to $r^{t-2}$, and any player who is unhappy after that move has load $M$ (and consequently wants to move to load $M$ or load $\leq M-2$). Repeating this argumentation shows that there can be no more than $t-3$ further deviations (leading to at most $2t = O(m)$ many deviations in the while-loop in total). □

The proof of Theorem 10 also yields the following upper bound on the running time of Algorithm 1.

Corollary 11. *For $\alpha = K \approx 1.1974$, the running time of Algorithm 1 can be bounded by* $\min\{O(n^2m), O(nm^2)\}$.

*Proof.* We first argue that each iteration of the while-loop can be implemented in $O(m)$. Note that the deviation cost $c_r(r, x_{-i})$ if some player $i$ deviates to $r$ (w.r.t. $x$) does not only depend on $r$, but also on player $i$'s strategy $x_i$ (at least in some cases, cf. Observation 7). However, it can only be different for two players if one of these players is using a resource with maximum load $M := M(x)$, and the other not. For that reason, we define for each resource $r \in R$ the costs $c_r^1(x)$ and $c_r^2(x)$ if a player with load $M$ and a player with load $< M$ deviates to it (for example, if $\ell_r(x) = M-1$ then $c_r^1(x) := a_r(\ell_r(x)+1) + \frac{B}{|M^{-1}(x)|}$ and $c_r^2(x) := a_r(\ell_r(x)+1) + \frac{B}{|M^{-1}(x)|+1}$). With $c^1(x) := \min_{r \in R} c_r^1(x)$ and $c^2(x) := \min_{r \in R} c_r^2(x)$, as well as $r^1(x)$ and $r^2(x)$ being the resources with smallest index in $\arg\min_{r \in R} c_r^1(x)$ and $\arg\min_{r \in R} c_r^2(x)$, we get that the players on resource $r' \in R$ are happy

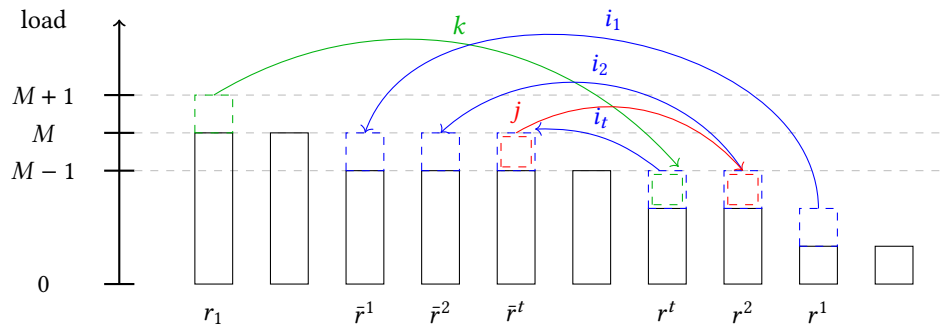



Fig. 10. Illustration for the case $\bar{r}^t \longrightarrow r^{t-1}$ (with $t = 3$).

if and only if $c_{r'}(x) \leq \alpha c^1(x)$ for $\ell_{r'}(x) = M$, and $c_{r'}(x) \leq \alpha c^2(x)$ for $\ell_{r'}(x) < M$. Furthermore, in case the players on resource $r'$ are unhappy, $r^1(x)$ (or $r^2(x)$, respectively) is a best response with smallest index for $\ell_{r'}(x) = M$ (for $\ell_{r'}(x) < M$). With this, it is easy to see that each iteration of the while-loop can be implemented in $O(m)$. Furthermore, the proof of Theorem 10 shows that in the $k$th iteration of the for-loop, the number of iterations of the while-loop can be bounded by $O(m)$, or alternatively also by $2k$ (no player moves more than twice). Since there are $n$ iterations of the for-loop and $\sum_{k=1}^{n} 2k = O(n^2)$, we get $\min\{O(n^2), O(nm)\}$ as an upper bound for the total number of iterations of the while-loop, implying the desired bound on the total running time. □

The proof of Theorem 10 also reveals that $n \geq 5$ and $m \geq 3$ need to hold for any instance which has no exact PNE, or, in other words, the following corollary holds.

Corollary 12. *Exact PNE always exist if $n \leq 4$ or $m \leq 2$.*

*Proof.* Note that the only case where the proof of Theorem 10 fails if we consider $\alpha = 1$ instead of $\alpha = K$ is the case corresponding to Figure 9. Since $r_1, \bar{r}^t$ and $r^t$ are three different resources, and with respect to $x'$ there are at least three players using $r_1$, at least one player using $\bar{r}^t$ and at least one player using $r^t$, the statement follows. □

## 4.3 Tightness of $\alpha = K$

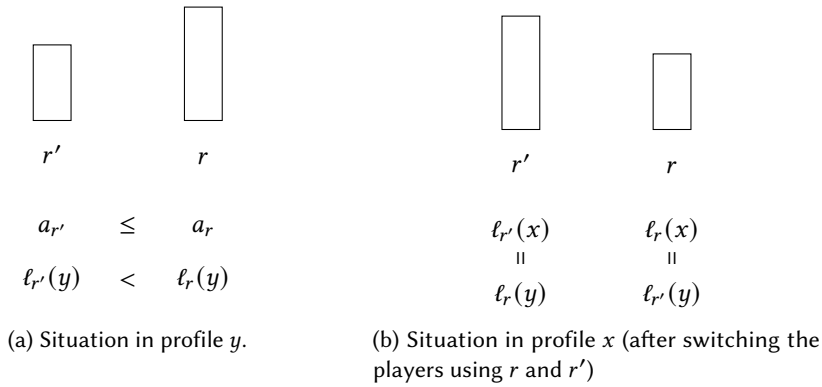
In this subsection, we provide an instance where no $\alpha$-approximate PNE with $\alpha < K$ exists, showing that $\alpha = K$ is the smallest possible value such that the existence of an $\alpha$-approximate equilibrium can be guaranteed.

The following lemma tells us that if we want to investigate the existence of an $\alpha$-PNE (for some $\alpha \in [1, 2]$), it suffices to consider profiles with decreasing loads.

Lemma 13. *Assume that the resource set $R = \{r_1, \ldots, r_m\}$ is ordered such that $a_1 \leq \cdots \leq a_m$. Then: For any $\alpha \in [1, 2]$, if there exists an $\alpha$-approximate PNE, then there also exists an $\alpha$-approximate PNE $x$ with decreasing loads, that is, with $\ell_1(x) \geq \cdots \geq \ell_m(x)$.*

*Proof.* Consider an $\alpha$-approximate PNE $y$. If the loads are decreasing we are done, thus assume that there are resources $r, r' \in R$ with $a_{r'} \leq a_r$ and $\ell_{r'}(y) < \ell_r(y)$; see Figure 11(a) for illustration. Consider the profile $x$ resulting from $y$ by switching the players using $r$ and $r'$, thus $\ell_r(x) = \ell_{r'}(y)$, $\ell_{r'}(x) = \ell_r(y)$ and $\ell_{\tilde{r}}(x) = \ell_{\tilde{r}}(y)$ for all $\tilde{r} \in R \setminus \{r, r'\}$; see Figure 11(b). We now show that $x$ is an $\alpha$-approximate PNE, too, which proves the

lemma. Let $M := M(y) = \max\{\ell_{\tilde{r}}(y) : \tilde{r} \in R\}$ denote the maximum load with respect to $y$. Note that $M = M(x)$ and $|M^{-1}(x)| = |M^{-1}(y)|$. We now show that all players are happy in $x$.




(a) Situation in profile $y$.

(b) Situation in profile $x$ (after switching the players using $r$ and $r'$)

Fig. 11. Illustration for Lemma 13.

First consider a player $j$ with $x_j \notin \{r, r'\}$. Clearly, the cost experienced by player $j$ did not change (comparing $y$ and $x$). Also, the deviation cost of player $j$ to $r'' \notin \{r, r'\}$ did not change. Furthermore, since the load on $r'$ increased, the deviation cost to $r'$ can only be larger than before (that is, w.r.t. $y$). Finally, the deviation cost to $r$ w.r.t. $x$ can only be larger than the deviation cost to $r'$ w.r.t. $y$. Using that player $j$ was happy in $y$, we conclude that the same holds in $x$.

Now assume player $j$ uses $r'$, that is, $x_j = r'$ and $y_j = r$. Since $\ell_{r'}(x) = \ell_r(y)$, the cost experienced by player $j$ can only be smaller in $x$ (compared to $y$). Furthermore, the deviation cost if player $j$ deviates to a resource $r'' \neq r$ did not change. Finally, the deviation cost of player $j$ to $r$ w.r.t. $x$ can only be larger than the deviation cost of player $j$ to $r'$ w.r.t. $y$. Since player $j$ was happy in $y$, we again conclude that player $j$ is also happy in $x$.

It remains to consider the players using resource $r$. Since the load on $r$ decreased, the cost experienced on $r$ can only be smaller in $x$ compared to $y$. It is furthermore easy to see that the deviation cost if a player deviates from $r$ to $r'$ can only be larger in $x$ than in $y$. This shows that players do not want to deviate from $r$ to $r'$ (w.r.t. $x$). To complete the proof, it remains to show that the same holds for a deviation from $r$ to $\bar{r} \neq r'$. If the deviation cost from $r$ to $\bar{r}$ did not change we are done, thus assume it changed. This is only possible in two cases: Either $\ell_{\bar{r}}(y) = M - 1$ and $\ell_r(y) = M$, or $\ell_{\bar{r}}(y) = M - 2$ and $r$ was the only maximum load resource in $y$. Note that in both cases, $\ell_r(x) = \ell_{r'}(y) \leq M - 1$.

Consider first the case that $\ell_{\bar{r}}(y) = M - 1$ and $\ell_r(y) = M$ and assume, by contradiction, that a deviation from $r$ to $\bar{r}$ is $\alpha$-improving in $x$, and thus

$$a_r(M-1) \geq a_r \ell_r(x) = c_r(x) > \alpha(a_{\bar{r}} M + \tfrac{B}{p+1}), \tag{8}$$

where $p := |M^{-1}(y)| \geq 1$ denotes the number of resources with load $M$ in $y$. On the other hand, using that $y$ is an $\alpha$-PNE,

$$c_r(y) = a_r M + \tfrac{B}{p} \leq \alpha(a_{\bar{r}} M + \tfrac{B}{p}). \tag{9}$$

Combining inequalities (8) and (9) leads to

$$\frac{a_r M + \frac{B}{p}}{\alpha} - \frac{B}{p} \leq a_{\bar{r}} M < \frac{a_r(M-1)}{\alpha} - \frac{B}{p+1}.$$

Multiplying with $\alpha \cdot p$ yields

$$a_r Mp + B - \alpha B < a_r(M-1)p - \alpha B \tfrac{p}{p+1} \iff (1 - \alpha + \alpha \tfrac{p}{p+1})B < -a_r p \leq 0.$$

Since $B > 0$ we get $1 - \alpha + \alpha \frac{p}{p+1} < 0$, or, equivalently, $\alpha > p + 1 \geq 2$. But this contradicts $\alpha \leq 2$.

To complete the proof, it remains to consider the case that $\ell_{\bar{r}}(y) = M - 2$ and $r$ is the only resource with load $M$ in $y$. As above, we assume by contradiction that

$$a_r(M-1) \geq a_r \ell_r(x) = c_r(x) > \alpha(a_{\bar{r}}(M-1)). \tag{10}$$

Since $y$ is an $\alpha$-PNE, we have

$$c_r(y) = a_r M + B \leq \alpha(a_{\bar{r}}(M-1) + \tfrac{B}{q+2}), \tag{11}$$

where $q \geq 0$ denotes the number of resources with load $M - 1$ in $y$. Combining inequalities (10) and (11) yields

$$\tfrac{a_r M + B}{\alpha} - \tfrac{B}{q+2} + a_{\bar{r}} \leq a_{\bar{r}} M < \tfrac{a_r(M-1)}{\alpha} + a_{\bar{r}},$$

and this implies

$$B(1 - \tfrac{\alpha}{q+2}) < -a_r \leq 0.$$

But with $B > 0$ and $1 - \frac{\alpha}{q+2} \geq 0$ (note that $\alpha \leq 2$), we get a contradiction. We have thus shown that players on $r$ do not want to deviate to $\bar{r} \neq r'$ (w.r.t. $x$), completing the proof. □

Using the above lemma, we can show the following theorem.

Theorem 14. *There exists an instance with three resources and five players such that there is no $\alpha$-approximate PNE for any $\alpha < K$, where $K \approx 1.1974$ is as specified in (3).*

*Proof.* Consider the instance with $m = 3$ resources, $n = 5$ players, budget $B = 1$, and resource cost coefficients

$$a_1 = 0,\ a_2 = K/2 - 1/4 \approx 0.3487,\ \text{and } a_3 = 1/K \approx 0.8351.$$

We proceed to show that there is no $\alpha$-approximate PNE for $\alpha < K$. Using $a_1 < a_2 < a_3$ and Lemma 13, it suffices to show that there is no $\alpha$-PNE among the five load profiles $(5, 0, 0)$, $(4, 1, 0)$, $(3, 2, 0)$, $(3, 1, 1)$, $(2, 2, 1)$. Let $\alpha < K$. We show that for any of these load profiles, there exists an $\alpha$-improving deviation, showing the claim. To this end, consider Table 5, where we provide a deviation for each candidate profile which is $\alpha$-improving if the given conditions on $\alpha$ are satisfied. It is easy to check that these conditions are indeed fulfilled for $\alpha < K$. □

## 5 Computing Optimal Approximate Equilibria

As in the last section, we again assume throughout this section that the underlying congestion game is a symmetric singleton congestion game, that is, each player chooses exactly one of the available resources.

In the last section, we showed that $\alpha = K$ is the smallest possible value for $\alpha$ such that the existence of an $\alpha$-approximate PNE can be guaranteed for *any* such instance of a multi-leader congestion game with an adversary. However, there are clearly instances where $\alpha$-PNE with $\alpha < K$ exist (in particular, all instances exhibiting an exact PNE). We show in this section how to compute efficiently a best approximate PNE *for a given instance*, that is, with smallest possible $\alpha$ such that an $\alpha$-PNE exists for *this* instance.

To this end, consider a multi-leader congestion game with an adversary with resource set $R = [m]$ and $a_1 \leq \cdots \leq a_m$.[2] We first show that for a given $\alpha \in [1, K]$, one can decide in polynomial time whether there exists an $\alpha$-PNE $x$. (Note that we can assume $\alpha \in [1, K]$ since we want to find the smallest possible $\alpha$.) Secondly, we argue that we only need to consider polynomially many values for $\alpha$.

[2]Note that, in this section, we use the simplified notation $R = [m]$ for the set of resources.

Table 5. $\alpha$-improving deviations for the candidate profiles.

| load profile | $\alpha$-improving deviation (of a player using $r$ to $r'$, notation $r \to r'$) | conditions on $\alpha$ |
|---|---|---|
| $(5,0,0)$ | $r_1 \to r_2$ | $5a_1 + B > \alpha a_2 \Leftrightarrow \alpha < 1/a_2 = \frac{2}{K-1/2} \approx 2.86$ |
| $(4,1,0)$ | $r_1 \to r_2$ | $4a_1 + B > \alpha 2a_2 \Leftrightarrow \alpha < 1/(2a_2) = \frac{1}{K-1/2} \approx 1.43$ |
| $(3,2,0)$ | $r_1 \to r_3$ | $3a_1 + B > \alpha a_3 \Leftrightarrow \alpha < 1/a_3 = K$ |
| $(3,1,1)$ | $r_3 \to r_2$ | $a_3 > \alpha 2a_2 \Leftrightarrow \alpha < \frac{1}{K(K-1/2)} = K$ |
| $(2,2,1)$ | $r_2 \to r_1$ | $2a_2 + B/2 > \alpha(3a_1 + B) \Leftrightarrow \alpha < 2a_2 + 1/2 = K$ |

Let $\alpha \in [1, K]$. We want to investigate the existence of an $\alpha$-PNE. Note that by Lemma 13, we can restrict our attention to profiles $x$ with decreasing loads, that is, with $\ell_1(x) \geq \cdots \geq \ell_m(x)$. Together with $a_1 \leq \cdots \leq a_m$, this leads to the following three easy observations (compare also Observation 7): Firstly, if several resources have the same load, then all players using these resources are happy if and only if the players on the last such resource are happy.[3] Secondly, deviating to the first among several resources having the same load is always cheapest among those resources. Thirdly, it is never beneficial to deviate to a resource where the load is smaller by one than the currently experienced load. Making use of these observations, it is easy to check whether there is an $\alpha$-PNE among the profiles with a difference of *at most two* between the largest and the smallest load: In the following three cases, we first test the profile in which all resources have the same load, that is, the difference between the largest and the smallest load is zero, then all profiles with difference between the largest and the smallest load equal to one, and finally the profiles where the difference is two.

(1) If $\frac{n}{m} \in \mathbb{N}$ and $a_m \cdot \frac{n}{m} + \frac{B}{m} \leq \alpha \cdot (a_1(\frac{n}{m} + 1) + B)$, any $x$ with $\ell_r(x) = \frac{n}{m}$ for all $r \in [m]$ is an $\alpha$-PNE.

(2) For each $k \in \{1, \ldots, m-1\}$ with $M := \frac{n-k+m}{m} \in \mathbb{N}$ and $M \geq 1$: If the following conditions are fulfilled, any $x$ with $\ell_r(x) = M$ for all $r \leq k$ and $\ell_r(x) = M - 1$ for all $r \in \{k+1, \ldots, m\}$ is an $\alpha$-PNE:

$$\begin{array}{rll} & a_m(M-1) & \leq \alpha \cdot (a_1(M+1) + B) \\ \text{if } k < m-1: & a_m(M-1) & \leq \alpha \cdot (a_{k+1} \cdot M + \frac{B}{k+1}) \end{array} \left.\vphantom{\begin{array}{l}a\\a\end{array}}\right\} \begin{array}{l}\text{players with load } M-1 \\ \text{happy}\end{array}$$
$$\text{if } k \geq 2: \; a_k \cdot M + \frac{B}{k} \leq \alpha \cdot (a_1(M+1) + B) \;\Big\}\; \text{players with load } M \text{ happy}$$

(3) For each $k \in \{1, \ldots, m-1\}$ and $k' \in \{k+1, \ldots, m\}$ with $M := \frac{n+2m+1-k-k'}{m} \in \mathbb{N}$ and $M \geq 2$: If the following conditions are fulfilled, any $x$ with $\ell_r(x) = M$ for all $r \leq k$, $\ell_r(x) = M - 1$ for all $r \in \{k+1, \ldots, k'-1\}$ and $\ell_r(x) = M - 2$ for all $r \in \{k', \ldots, m\}$ is an $\alpha$-PNE. (Note that $k' = k + 1$ is possible, which means that there

[3]Expressions such as *first* or *last* are with respect to the ordering $r_1, r_2, \ldots, r_m$ of the resources.

are no resources having load $M-1$.)

$$\left.\begin{array}{rl} a_m(M-2) & \leq \alpha \cdot (a_1(M+1)+B) \\ \text{if } k' > k+1: \ a_m(M-2) & \leq \alpha \cdot (a_{k+1} \cdot M + \frac{B}{k+1}) \\ \text{if } k' < m: \ a_m(M-2) & \leq \alpha \cdot (a_{k'}(M-1)) \end{array}\right\} \begin{array}{l}\text{players with load } M-2 \\ \text{happy}\end{array}$$

$$\left.\begin{array}{rl} \text{if } k' > k+1: \ a_{k'-1}(M-1) & \leq \alpha \cdot (a_1(M+1)+B) \\ \text{if } k' > k+2: \ a_{k'-1}(M-1) & \leq \alpha \cdot (a_{k+1} \cdot M + \frac{B}{k+1}) \end{array}\right\} \begin{array}{l}\text{players with load } M-1 \\ \text{happy}\end{array}$$

$$\left.\begin{array}{rl} \text{if } k = 1: \ a_k \cdot M + B & \leq \alpha \cdot (a_{k'}(M-1) + \frac{B}{k'}) \\ \text{if } k \geq 2: \ a_k \cdot M + \frac{B}{k} & \leq \alpha \cdot (a_1(M+1)+B) \\ \text{if } k \geq 2: \ a_k \cdot M + \frac{B}{k} & \leq \alpha \cdot (a_{k'}(M-1)) \end{array}\right\} \begin{array}{l}\text{players with load } M \\ \text{happy}\end{array}$$

Clearly, one can check the above three cases in $O(m^2)$, leading to the following result.

Lemma 15. *We can decide in $O(m^2)$ whether there is an $\alpha$-PNE among the profiles with a difference of at most two between the largest and the smallest load. In case of existence, we can also compute a corresponding load vector efficiently.*

In the following, we analyze the profiles where the difference between the largest and the smallest load is *larger than two*. For such a profile $x$, we introduce the following notation. Let $M := M(x)$ denote the maximum load in $x$. Then, $k = k(x)$ denotes the largest resource having load $M$, $k' = k'(x)$ is the smallest resource with load strictly smaller than $M-1$, and $k'' = k''(x)$ denotes the smallest resource with load strictly smaller than $M-2$. In other words, $\ell_r(x) = M$ for all $r \in \{1, \ldots, k\}$, $\ell_r(x) = M-1$ for all $r \in \{k+1, \ldots, k'-1\}$, $\ell_r(x) = M-2$ for all $r \in \{k', \ldots, k''-1\}$, and $\ell_r(x) \leq M-3$ for all $r \in \{k'', \ldots, m\}$, see Figure 12 for illustration. Note that $k' = k+1$ or $k' = k''$ are possible, in which cases there are no resources with load $M-1$ or $M-2$, respectively.

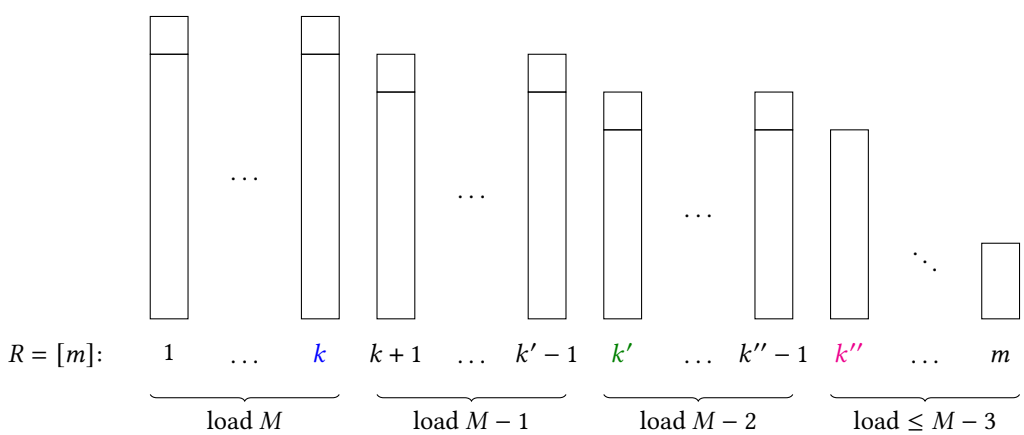



Fig. 12. Illustration for the definition of $k$, $k'$, and $k''$.

We now argue that there exist values $\bar{c}_M = \bar{c}_M(x)$ and $\bar{c}_{<M} = \bar{c}_{<M}(x)$ such that $x$ is an $\alpha$-PNE if and only if

$$a_k \cdot M + B/k \leq \alpha \cdot \bar{c}_M \quad \text{and} \tag{12}$$

$$a_r \cdot \ell_r(x) \leq \alpha \cdot \bar{c}_{<M} \text{ for all } r > k. \tag{13}$$

Essentially, the values $\bar{c}_M$ and $\bar{c}_{<M}$ describe the cost of a best response for a player using a resource with load $M$, and load smaller than $M$, respectively. We now explain this in more detail. Consider, for example, a player $i$ using

a resource with load $< M$. Then, the cost of a best response for player $i$ is

$$\begin{aligned}&\min\{\pi_i(r, x_{-i}) : r \in R, r \neq x_i\}\\ =&\min\{a_1 \cdot (M+1) + B, a_{k+1} \cdot M + \tfrac{B}{k+1} \text{ if } \ell_{k+1}(x) = M-1 \text{ and } k+1 \neq x_i,\\ &\quad a_r \cdot (\ell_r(x)+1)\ \forall\, r \in R \text{ with } \ell_r(x) \leq M-2 \text{ and } r \neq x_i\}.\end{aligned}$$

Using this, it is easy to see that $a_{x_i}\ell_{x_i}(x) \leq \alpha \cdot \min\{\pi_i(r, x_{-i}) : r \in R, r \neq x_i\}$ if and only if

$$\begin{aligned}a_{x_i}\ell_{x_i}(x) &\leq \alpha \cdot \min\{a_1 \cdot (M+1) + B, a_{k+1} \cdot M + \tfrac{B}{k+1} \text{ if } \ell_{k+1}(x) = M-1,\\ &\qquad a_r \cdot (\ell_r(x)+1)\ \forall\, r \in R \text{ with } \ell_r(x) \leq M-2\}\\ &= \alpha \cdot \min\{a_1 \cdot (M+1) + B, a_{k+1} \cdot M + \tfrac{B}{k+1} \text{ if } k' \geq k+2,\\ &\qquad a_{k'} \cdot (M-1) \text{ if } k' < k'', a_r \cdot (\ell_r(x)+1)\ \forall\, r \geq k''\}\\ &=: \alpha \cdot \bar{c}_{<M}.\end{aligned}$$

By very similar argumentation, one can derive that

$$\bar{c}_M := \begin{cases}\min\{a_{k'} \cdot (M-1) + B/k' \text{ if } k' < k'', a_r \cdot (\ell_r(x)+1)\ \forall\, r \geq k''\}, & \text{if } k = 1,\\ \min\{a_1 \cdot (M+1) + B, a_{k'} \cdot (M-1) \text{ if } k' < k'', a_r \cdot (\ell_r(x)+1)\ \forall\, r \geq k''\}, & \text{if } k \geq 2.\end{cases}$$

Using the introduced notation, we now describe how to check the existence of an $\alpha$-PNE among all profiles with a difference larger than two between the largest and the smallest load. We start with Lemma 16 below, which follows from two main insights (see the proof for all details): First of all, for given values of $M, k, k'$ and $k''$, there are only polynomially many possible values for $\bar{c}_M$ and $\bar{c}_{<M}$. Secondly, if we (only) know the values $M, k, k'$ and $k''$ as well as $\bar{c}_M$ and $\bar{c}_{<M}$ of some $\alpha$-PNE $x$, we can reconstruct in polynomial time the corresponding load vector $\ell(x)$. To this end, note that the minimum properties of $\bar{c}_M$ and $\bar{c}_{<M}$, as well as the Nash condition (13), yield upper and lower bounds for the loads $\ell_r(x)$ of all resources $r \geq k''$ (for $r < k''$, the load $\ell_r(x)$ is uniquely determined by the definitions of $M, k, k'$ and $k''$).

Lemma 16. *Given $M \in \{\max\{3, \lceil \frac{n}{m} \rceil\}, \ldots, n\}$, $k \in \{1, \ldots, m-1\}$, $k' \in \{k+1, \ldots, m\}$ and $k'' \in \{k', \ldots, m\}$, we can decide efficiently if there exists an $\alpha$-approximate PNE $x$ such that $M(x) = M$, $k(x) = k$, $k'(x) = k'$ and $k''(x) = k''$. In case of existence, we can also compute a corresponding load vector efficiently.*

*Proof.* Assume for the moment that there exists an $\alpha$-PNE $x$ with $M(x) = M$, $k(x) = k$, $k'(x) = k'$, and $k''(x) = k''$. Recall that

$$\begin{aligned}\bar{c}_{<M}(x) := \min\{&a_1(M+1) + B, a_{k+1}M + \tfrac{B}{k+1} \text{ if } k' \geq k+2, a_{k'}(M-1) \text{ if } k' < k'',\\ &a_r(\ell_r(x)+1)\ \forall\, r \geq k''\}\end{aligned}$$

and

$$\bar{c}_M(x) := \begin{cases}\min\{a_{k'}(M-1) + B/k' \text{ if } k' < k'', a_r(\ell_r(x)+1)\ \forall\, r \geq k''\}, & \text{if } k = 1,\\ \min\{a_1(M+1) + B, a_{k'}(M-1) \text{ if } k' < k'', a_r(\ell_r(x)+1)\ \forall\, r \geq k''\}, & \text{if } k \geq 2.\end{cases}$$

Since we know the values $M, k, k', k''$, and for $r \geq k''$ the load $\ell_r(x)$ attains values in $\{0, 1, \ldots, M-3\}$, there are at most $3 + (m - k'' + 1) \cdot (M-2) = O(mn)$ many possible values that $\bar{c}_{<M}(x)$ can attain, and at most $2 + (m - k'' + 1) \cdot (M-2) = O(mn)$ many possible values that $\bar{c}_M(x)$ can attain. Thus we can identify polynomially large *candidate sets* $\bar{C}_M$ and $\bar{C}_{<M}$ which definitely contain $\bar{c}_M(x)$ and $\bar{c}_{<M}(x)$, respectively (for instance, for $k = 1$ and $k' < k''$ we define

$$\bar{C}_M := \{a_{k'}(M-1) + B/k', a_r(\ell+1)\ \forall\, r \geq k'' \text{ and } \forall\, 0 \leq \ell \leq M-3\},$$

and in the same manner for the other cases and for $\bar{C}_{<M}$). We further reduce these candidate sets by deleting all values $\bar{c}_M \in \bar{C}_M, \bar{c}_{<M} \in \bar{C}_{<M}$ not satisfying one or more of the following inequalities, where the first inequalities come from the Nash conditions (players using resources with load $M, M-1$ or $M-2$ do not want to deviate), and the remaining inequalities are due to the minimum properties of $\bar{c}_M(x)$ and $\bar{c}_{<M}(x)$, respectively:

1. : $a_k \cdot M + \frac{B}{k} \leq \alpha \cdot \bar{c}_M$;
   if $k' \geq k+2$: $a_{k'-1} \cdot (M-1) \leq \alpha \cdot \bar{c}_{<M}$;
   if $k' < k''$: $a_{k''-1} \cdot (M-2) \leq \alpha \cdot \bar{c}_{<M}$;
2. : if $k \geq 2$: $a_1 \cdot (M+1) + B \geq \bar{c}_M$;
   if $k = 1$ and $k' < k''$: $a_{k'} \cdot (M-1) + \frac{B}{k'} \geq \bar{c}_M$;
   if $k \geq 2$ and $k' < k''$: $a_{k'} \cdot (M-1) \geq \bar{c}_M$;
3. : $a_1 \cdot (M+1) + B \geq \bar{c}_{<M}$;
   if $k' \geq k+2$: $a_{k+1} \cdot M + \frac{B}{k+1} \geq \bar{c}_{<M}$;
   if $k' < k''$: $a_{k'} \cdot (M-1) \geq \bar{c}_{<M}$.

Note that for all $r \geq k''$ we have by the Nash conditions that

$$a_r \ell_r(x) \leq \alpha \cdot \bar{c}_{<M}(x) \iff \ell_r(x) \leq \left\lfloor \frac{\alpha \cdot \bar{c}_{<M}(x)}{a_r} \right\rfloor.^{4}$$

Furthermore, $\ell_r(x) \leq M-3$ (by definition, $r = k''$ is the smallest resource with load strictly smaller than $M-2$). On the other hand, we also get a lower bound on $\ell_r(x)$ by using the minimum properties of $\bar{c}_M(x)$ and $\bar{c}_{<M}(x)$, since

$$a_r(\ell_r(x)+1) \geq \bar{c}_M(x) \iff \ell_r(x) \geq \left\lceil \frac{\bar{c}_M(x)}{a_r} \right\rceil - 1$$

and

$$a_r(\ell_r(x)+1) \geq \bar{c}_{<M}(x) \iff \ell_r(x) \geq \left\lceil \frac{\bar{c}_{<M}(x)}{a_r} \right\rceil - 1.^{5}$$

Altogether, these observations lead to Algorithm 2, which is clearly a polynomial algorithm (its running time can be bounded by $O(m^3n^2)$ since we have at most $O(m^2n^2)$ iterations of the loop in line 10 and one of these iterations takes $O(m)$). For correctness, assume that it terminates in line 26 with a load vector $\ell$ and pair $(\bar{c}_M, \bar{c}_{<M})$. We need to show that there exists an $\alpha$-PNE $x$ with $\ell(x) = \ell$, $M(x) = M$, $k(x) = k$, $k'(x) = k'$, and $k''(x) = k''$. Clearly, $\ell_r \geq 0$ for all $r \in R$ and $\sum_{r \in R} \ell_r = n$. Thus there exists a strategy profile $x$ with load vector $\ell(x) = \ell$, which directly implies $M(x) = M$, $k(x) = k$, $k'(x) = k'$ and $k''(x) = k''$. It remains to show that $x$ is in fact an $\alpha$-PNE. We obtain that $a_k \cdot M + B/k \leq \alpha \bar{c}_M$ and $a_r \ell_r(x) \leq \alpha \bar{c}_{<M}$ for all $r > k$ hold (by computation of the candidate sets $\bar{C}_M, \bar{C}_{<M}$, and upper bounds $b'_r$). Furthermore, $\bar{c}_M \leq \bar{c}_M(x)$ and $\bar{c}_{<M} \leq \bar{c}_{<M}(x)$ (by computation of the candidate sets $\bar{C}_M, \bar{C}_{<M}$, and lower bounds $b_r$). Altogether, this shows that $x$ is an $\alpha$-PNE, and that the algorithm is correct (clearly, if it terminates in a line different from line 26, there cannot be an $\alpha$-PNE with the desired properties). □

Since there are only polynomially many possible values that $M, k, k'$, and $k''$ can attain, Lemma 16 leads to an efficient procedure for testing existence of an $\alpha$-PNE among the profiles with a difference larger than two between largest and smallest load. Using this, we can now show the main result of this section.

[4]Note here that we can assume w.l.o.g. that $a_r > 0$ for all $r \geq k''$, otherwise there is no $\alpha$-PNE corresponding to $M, k, k', k''$ since players on $r_1$ would always want to deviate to $r$.

[5]Note that $\bar{c}_{<M}(x) \leq \bar{c}_M(x)$ and thus the lower bound on $\ell_r(x)$ achieved from $\bar{c}_M(x)$ is stronger.

**Algorithm 2:** Subroutine needed to compute best-possible $\alpha$.

**Input:** $\alpha \in [1, K]$ and $M, k, k', k''$ as specified in Lemma 16.
**Output:** Load vector $\ell = \ell(x)$ corresponding to an $\alpha$-PNE $x$ with $M(x) = M$, $k(x) = k$, $k'(x) = k'$ and $k''(x) = k''$, if such an $x$ exists.

1 **if** $a_{k''} = 0$ **then**
2 Terminate the algorithm; `// No` $\alpha$`-PNE` $x$ `corresponding to` $M, k, k', k''$.
3 $\ell_r \leftarrow M \; \forall \, r \in \{1, \dots, k\}$;
4 $\ell_r \leftarrow M - 1 \; \forall \, r \in \{k+1, \dots, k'-1\}$;
5 $\ell_r \leftarrow M - 2 \; \forall \, r \in \{k', \dots, k''-1\}$;
6 $n' \leftarrow n - kM - (k' - k - 1)(M - 1) - (k'' - k')(M - 2)$; `// remaining players`
7 **if** $n' < 0$ **then**
8 Terminate the algorithm; `// No` $\alpha$`-PNE` $x$ `corresponding to` $M, k, k', k''$.
9 Compute candidate sets $\bar{C}_M, \bar{C}_{<M}$ for $\bar{c}_M, \bar{c}_{<M}$ as described in the proof of Lemma 16;
10 **forall** $(\bar{c}_M, \bar{c}_{<M}) \in \bar{C}_M \times \bar{C}_{<M}$ *with* $\bar{c}_M \geq \bar{c}_{<M}$ **do**
11 **for** $r = k'', \dots, m$ **do**
12 $b_r \leftarrow \max\{0, \lceil \frac{\bar{c}_M}{a_r} \rceil - 1\}$; `// lower bound on` $\ell_r$
13 $b'_r \leftarrow \min\{M - 3, \lfloor \frac{\alpha \cdot \bar{c}_{<M}}{a_r} \rfloor\}$; `// upper bound on` $\ell_r$
14 **if** $b_r > b'_r$ **then**
15 Go to Line 10 and continue with the next pair $(\bar{c}_M, \bar{c}_{<M})$, or terminate the algorithm if current pair was the last;
16 `// current pair not correct`
17 **if** $n' \notin [\sum_{r \geq k''} b_r, \sum_{r \geq k''} b'_r]$ **then**
18 Go to Line 10 and continue with the next pair $(\bar{c}_M, \bar{c}_{<M})$, or terminate the algorithm if current pair was the last;
19 `// current pair not correct`
20 $\ell_r \leftarrow b_r \; \forall \, r \in \{k'', \dots, m\}$;
21 $n' \leftarrow n' - \sum_{r \geq k''} b_r$;
22 **while** $n' > 0$ **do**
23 **for** $r = k'', \dots, m$ **do**
24 $\ell_r \leftarrow \min\{b'_r, b_r + n'\}$;
25 $n' \leftarrow n' - (\ell_r - b_r)$;
26 Terminate the algorithm;

Theorem 17. *For a given instance, we can efficiently compute the smallest possible $\alpha$ such that an $\alpha$-approximate PNE exists, as well as a corresponding load vector.*

*Proof.* First note that if $x$ is an $\alpha$-PNE corresponding to the best possible $\alpha$, there need to be resources $r, r' \in R$ (as well as a player $i$ with $x_i = r$) such that $a_r \ell_r(x) + \kappa^*_r(x) = \alpha \cdot (a_{r'} \ell_{r'}(r', x_{-i}) + \kappa^*_{r'}(r', x_{-i}))$ (otherwise $\alpha$ would not be the smallest possible approximation factor), or, equivalently,

$$\alpha = \frac{a_r \ell_r(x) + \kappa^*_r(x)}{a_{r'} \ell_{r'}(r', x_{-i}) + \kappa^*_{r'}(r', x_{-i})}.$$

Since there are only $O(nm^2)$ many possible values for $a_r \ell_r(x) + \kappa_r^*(x)$, as well as for $a_{r'} \ell_{r'}(r', x_{-i}) + \kappa_{r'}^*(r', x_{-i})$, there are only $O(n^2 m^4)$ many possible values for $\alpha$. Since for a fixed $\alpha$ we can decide existence (and compute a load vector) of an $\alpha$-PNE efficiently (by Lemmas 15 and 16), the theorem follows. □

## 6 Further Remarks and Conclusion

Before we conclude our paper, we briefly discuss a variation of the setting discussed in the preceding sections, namely additive (rather than multiplicative) approximate PNE.

### 6.1 Additively Approximate PNE

In addition to multiplicatively approximate equilibria, as studied in this paper, it is also possible to define *additively approximate equilibria*. An *$\varepsilon$-additive approximate PNE* is a strategy profile $x \in X$ such that

$$\pi_i(x) \leq \pi_i(y_i, x_{-i}) + \varepsilon \text{ for all } y_i \in X_i \text{ and all } i \in N.$$

Contrasting the case of $K$-multiplicative approximate PNE, however, the existence of $\varepsilon$-additive approximate PNE for the class of games studied in Sections 4 and 5 of this paper cannot be guaranteed for any constant $\varepsilon > 0$.

To see this, note that the considered multi-leader congestion games with an adversary are invariant under scaling in the sense that multiplying all cost coefficients $a_r$ and the adversary's budget $B$ with the same factor $\lambda > 0$ results in a game in which each player's private cost for any given strategy profile is scaled by the same factor $\lambda$. As a result, given any constant $\varepsilon > 0$, any instance of the game that does not have an exact PNE (such as, e.g., the one described in Example 5) can be scaled in such a way that it does not allow for an $\varepsilon$-additive approximate PNE.

We remark that, while the existence of approximate PNE with a small additive constant cannot be guaranteed, the approach in Section 5 can easily be adjusted to compute $\varepsilon$-additive approximate PNE with the smallest possible $\varepsilon$ for a given instance.

### 6.2 Conclusion

We introduced a multi-leader single-follower Stackelberg game in which the leaders play a congestion game and subsequently an adversary attacks the resources with maximum loads, causing additional costs for the leaders. We first showed that pure Nash equilibria always exist if the leaders' strategies are given by bases of matroids and the (linear) congestion cost functions are identical. However, if one of these two conditions is violated, PNE do not exist in general. Motivated by this, we studied approximate equilibria for the case of symmetric singleton congestion games and resource-specific linear congestion cost functions. Our main result shows that a $K$-approximate PNE always exists, where $K \approx 1.1974$ is the unique solution of a cubic polynomial equation. To this end, we presented an efficient algorithm which computes a $K$-approximate PNE. Furthermore, we showed that the factor $K$ is tight by providing an instance where no $\alpha$-approximate PNE with $\alpha < K$ exists. However, for a *specific* instance there might be a better $\alpha$-approximate PNE, that is, with $\alpha < K$. We presented an efficient procedure that computes a *best* approximate PNE of a given instance.

Our work suggests several interesting directions for further research regarding multi-leader congestion games with an adversary. For example, one could analyze whether the results from Section 4 continue to hold if one allows for more general strategy spaces in the leaders' congestion game, for example, bases of matroids (and resource-specific congestion cost functions). Another interesting generalization is to consider different cost functions. Regarding this, note that our results from Section 4 (and similarly in Section 5) crucially rely on the existence of an ordering of the resources such that the loads are always nonincreasing during the course of Algorithm 1. For affine congestion cost functions $a_r \ell_r(x) + b_r$, with different $b_r \geq 0$, or for resource-specific

attacker costs, there is in general no such ordering, thus it is not clear how to apply our technique. However if the mentioned ordering exists, which is for example the case if the congestion cost functions are monomials $a_r(\ell_r(x))^d$ for some $d > 1$ (the ordering then is such that the $a_r$s are nondecreasing), we believe that our algorithm (and its analysis) can be used, and it would be very interesting to analyze which approximation factor can be guaranteed in such a setting. It would furthermore be interesting to analyze the quality of approximate PNE. For example, one could measure the social cost of a strategy profile by the total cost of all players, and then compare a (best or worse) approximate PNE to a social optimum.

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